EASTM 37 (2013)/2014: 12-35

Wang Xiaotong on Right Triangles: Six Problems from 'Continuation of Ancient Mathematics' (Seventh Century AD)[1]

Tina Su-lyn Lim 林淑铃 and Donald B. Wagner 华道安

[Tina Su-lyn Lim is a senior application manager with NNIT (Novo Nordisk Information Technology) in Copenhagen, Denmark. This article formed part of her M.Sc. thesis in Mathematics at the University of Copenhagen, 2006. Contact: ltsulyn@gmail.com]

[Donald B. Wagner has taught at the University of Copenhagen, the University of Victoria (British Columbia), the Technical University of Berlin, and Sichuan University. He has written widely on the history of science and technology in China; his most recent book is the volume on Ferrous Metallurgy *in Joseph Needham's* Science and Civilisation in China*. Contact: dwag@alum.mit.edu]*

* * *

Abstract: Wang Xiaotong's 王孝通 *Jigu suanjing* 緝古算經 is primarily concerned with problems in solid and plane geometry leading to polynomial equations which are to be solved numerically using a procedure similar to Horner's Method. We translate and analyze here six problems in plane geometry. In each case the solution is derived using a dissection of a 3-dimensional object. We suggest an interpretation of one fragmentary comment which at first sight appears to refer to a dissection of a 4-dimensional object.

Wang Xiaotong (late sixth to seventh century AD, exact dates unknown) served the Sui and Tang dynasties in posts concerned with calendrical calculations, and presented his book, *Jigu suanshu* 緝古算術, 'Continuation of ancient mathematics', to the Imperial court at some time after AD 626. In 656 it was made one of ten official 'canons' (*jing* 經) for mathematical education, and was retitled *Jigu suanjing*.[2] The book contains 20 problems:

---

[1] We are grateful to Karine Chemla for drawing our attention to Wang Xiaotong's book, and for detailed comments on this article; to Jesper Lützen and Ivan Tafteberg for comments on an earlier version; and to Mark Elvin, Jia-ming Ying 英家銘, and four anonymous reviewers for additional comments.

[2] *Jiu Tang shu* 1975, 32: 1168, 47: 2039, 4: 76, 44: 1892; Qian Baocong 1963: 487.

one astronomical problem, then 13 on solid geometry, then six on right triangles. All but the first provide extensions of the methods in the mathematical classic *Jiuzhang suanshu* 九章算術 ('Arithmetic in nine chapters', perhaps first century AD): the solid-geometry methods in Chapter 5 and the right-triangle methods in Chapter 9,[3] thus 'continuing' ancient mathematics. All but the first require the extraction of a root of a cubic or (in two cases) quadratic equation. The present article is concerned with problems 15–20, on right triangles.

Each problem describes a geometric situation, states some given quantities, and asks for one or more other quantities. Then the answer is given, and finally an algorithm for computing the answer. Printed in smaller characters in the text are comments which generally give explanations of the algorithms of the main text. All commentators appear to agree that the comments are by Wang Xiaotong himself, but we have noticed some differences in terminology between the comments and the main text,[4] and feel therefore that the question of the authorship of the comments should be left open. Perhaps most but not all of the comments are by him.

The Text

The history of the text is discussed by the modern editors, Qian Baocong (1963: 490–491) and Guo Shuchun and Liu Dun (1998: 1: 21–22); see also He Shaogeng (1989). All extant editions of *Jigu suanjing* go back to an edition printed in the early thirteenth century AD. One hand-copy of this edition survived to the twentieth century; Qian Baocong appears to have seen it.[5] It is now lost, but in 1684 a copy was included in a collectaneum, *Jiguge congshu* 汲古閣叢書. This version, the oldest surviving text of the *Jigu suanjing*, is now available in facsimile on the World Wide Web.[6] The best

---

[3] Chemla and Guo 2004: 387–457, 661–745; see also Cullen 1993; Shen Kangshen et al. 1999: 250–306, 439–517.

[4] E.g. in the commentary to problem 3, Qian Baocong 1963: 506, lines 5–8. Here *bienao* 鼈臑 is written in full, while it is always abbreviated as *bie* in the main text; *yanchu* 羨除 is once written in full and twice abbreviated as *chu*, while in the main text it is always abbreviated as *yan*.

[5] Qian Baocong describes the physical condition of the manuscript, and refers to it throughout his critical edition. He Shaogeng (1989: 37) states that the manuscript is 'now' in the Palace Museum in Beijing, but he does not appear to have seen it himself. Guo Shuchun and Liu Dun state that it is now lost.

[6] This is the *Tianlu Linlang congshu* 天祿琳瑯叢書 edition, a facsimile printed in 1931, available at http://www.scribd.com/doc/78515037 (accessed 6 December 2013).

Qing critical edition is that of Li Huang (1832);[7] others are by Dai Zhen (1777), Bao Tingbo (1780), Zhang Dunren (1803), and the Korean mathematician Nam Pyŏng-Gil (1820–1869).[8] All of these are available on the Web.

The standard modern edition has long been Qian Baocong's (1963, 2: 487–527; important corrections, 1966), but that of Guo Shuchun and Liu Dun (1998) has much to recommend it. In the parts of the text treated in the present article there is no important difference between the two.

The 1684 edition is marred by numerous obvious scribal errors which must be corrected by reference to the mathematical context. Guo and Liu (1998, 1: 22) note that Li Huang introduced ca. 700 emendations to the text, and that Qian Baocong followed most of these but introduced 20 new emendations. The part of the text considered here gives special difficulties, for it appears that the last few pages of the Southern Song hand copy were damaged, and a great many characters are missing. The problems involved in reconstructing the text are considered further below.

The *Jigu suanjing* has not been much studied in modern times. The two critical editions, already mentioned, do not explicitly comment on the mathematical content. Lin Yanquan (2001) translates the text into modern Chinese, expresses the calculations in modern notation, and gives derivations of some of the formulas. Deeper studies of individual parts of the text are by Shen Kangshen (1964), Qian Baocong (1966), He Shaogeng (1989), Wang Rongbin (1990), Guo Shiying (1994), and Andrea Bréard (1999: 95–99, 333–336, 353–356; 2002). The derivations of Problems 15 and 17 (Figures 2 and 3–4 below) have been reconstructed by Lin Yanquan, by He Shaogeng, and by Wang Rongbin; our reconstruction of the derivation of Problem 19 (Figure 5) is new.

The comments in the text give special difficulties, for two reasons. Being written in smaller characters, they are more subject to banal scribal errors, and their content is more abstract and complex than the main text. They have hardly been studied at all by modern scholars: the most recent serious study we have found is that of Luo Tengfeng (1770–1841) [1993].[9]

---

[7] On Li Huang's edition see Huang Juncai 2008.

[8] On Nam Pyŏng-Gil see Jia-ming Ying 2011; Horng Wann-sheng 2002; Zhang Fukai 2005.

[9] The only exceptions to this statement appear to be the highly speculative reconstructions of the fragmentary commentary to Problem 17 by He Shaogeng (1989) and Wang Rongbin (1990).

Solving cubic equations by 'Horner's method'

In the problem solutions given by Wang Xiaotong the coefficients of a cubic equation are calculated, after which the reader is instructed to 'extract the cube root'. Wang Xiaotong gives no indication of how this was done, but it was obviously a well-known procedure, for algorithms are described in detail in the *Jiuzhang suanshu* and the *Zhang Qiujian suanjing* 張邱建算經 (fifth century AD) for the special case of extracting a cube root (the cubic equation $x^3 = A$), and extensions of the latter algorithm to general polynomials of all orders are given in several Chinese books from after Wang Xiaotong's time.[10]

So much has been written about Chinese methods for extracting the roots of polynomial equations that it is not necessary to go into detail here. Let it suffice here to say that the algorithm is equivalent to 'Horner's method' as sometimes taught in modern schools and colleges.[11] The first digit of the root is determined, the roots of the equation are reduced by the value of that digit (an operation which amounts to a change of variable), the next digit is determined, and this step is repeated until the desired precision is reached. The operation is carried out on paper in the Western version, in the Chinese version with 'calculating rods' laid out on a table.

Wang Xiaotong's text describes how each of the elements of the cubic equation is calculated, using the following terminology:

shi 實, the constant term
fangfa 方法, the linear coefficient
lianfa 廉法, the quadratic coefficient

The cubic coefficient is always 1, and is never mentioned.[12]

---

10 Chemla & Guo 2004: 322–335, 363–379; Shen et al. 1999: 175–195, 204–226; Wang & Needham 1955; Lam Lay Yong 1970; 1977: 195–196, 251–285; 1986; Libbrecht 1973: 175–191; Chemla 1994; Martzloff 1997: 221–249.

11 See e.g. Rees & Sparks 1967: 294–297 as well as numerous pages on the World Wide Web. Horner (1819) presented a procedure for approximating roots of any infinitely differentiable function, but modern descriptions of 'Horner's method' consider only the special case of polynomial functions.

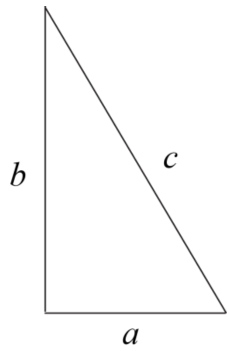
12 Commentators are divided on the function of the word *cong* / *zong* 從 in the phrase 從開立方除. Qian Baocong (1966: 46–47) reviews the various attempts to explain it, and comes to the conclusion that in each case it belongs at the end of the previous sentence, and means 'follow, accompany' (*gensui* 跟随). This is difficult for us to understand, and we omit the word in our translation. Another interpretation is given by Chemla and Guo (2004: 65, 912).

The triangle problems

The Chinese text of problems 15–20 and our translation are given in the final section of this article. Here we give an overview using modern terminology.

In pre-modern Chinese mathematics the sides of a right triangle are referred to as *gou* 句, 'shorter leg', *gu* 股, 'longer leg', and *xian* 弦, 'hypotenuse'. We translate these terms as 'base', 'leg', and 'hypotenuse', and in equations denote them as *a*, *b*, and *c* respectively; see Figure 1.

Figure 1



A peculiarity of the six triangle problems is that no units are given for the quantities involved. This is virtually unique in all of pre-modern Chinese mathematics (including Wang Xiaotong's Problems 1–14), in which problems are normally stated with reference to practical situations.

In **Problem 15** the given quantities are:

$$ab = 706\ {}^{1}/_{50}$$

$$c-a = 36\ {}^{9}/_{10}$$

and the values of *a*, *b*, and *c* are required. In the solution the coefficients of a cubic equation are calculated:

$$a^3 + \frac{(c-a)}{2}a^2 = \frac{(ab)^2}{2(c-a)} \qquad (1)$$

$$a^3 + 18\,{}^{9}\!/_{20}\,a^2 = 6754\,{}^{129}\!/_{500}$$

This has one real root,

$$a = 14\ {}^{7}/_{20}$$

whereafter it is straightforward to calculate the remaining quantities:

$$b = \frac{ab}{a} = 49\ {}^{1}/_{5}$$

$$c = a + (c - a) = 51\ {}^{1}/_{4}$$

A comment, printed in smaller characters, explains (1) in a mixture of algebraic and geometric reasoning. It first notes that

$$(ab)^2 = a^2b^2$$

The comment refers to a *fang* 方 (rectangular parallelepiped) and to quantities 'lined up'; this suggests a geometric rather than algebraic derivation of the method. We reconstruct this derivation as in Figure 2. We note that

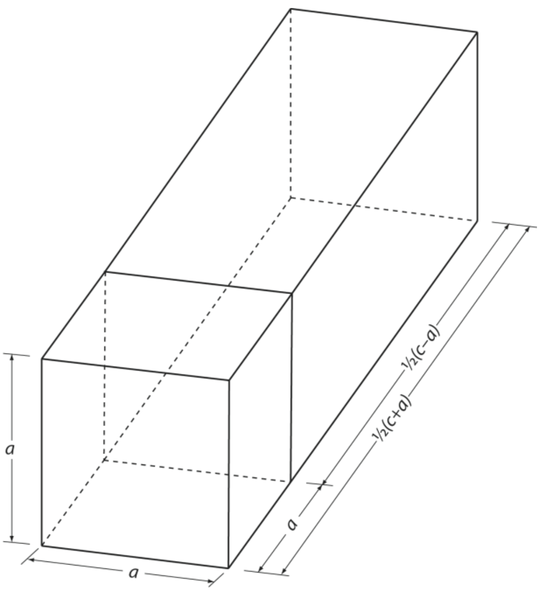
$$\frac{a^2b^2}{2(c-a)} = \frac{a^2\left(c^2-a^2\right)}{2(c-a)} = a^2\,\frac{a+c}{2}$$

(which Wang Xiaotong does not state explicitly). Then using Figure 2,

$$\frac{(ab)^2}{2(c-a)} = a^3 + \left(\frac{c-a}{2}\right)a^2$$

It is not known whether Wang Xiaotong's book originally included illustrations like Figure 2. We think it probably did not; he seems simply to give verbal descriptions of the geometric constructions which he uses.

Figure 2



Problem 16 is equivalent to Problem 15. The given quantities are

$$ab = 4036\ {}^{1}/_{5}$$

$$c–b = 6\ {}^{1}/_{5}$$

and c is required. The cubic equation here is

$$b^3 + \frac{c-b}{2} b^2 = \frac{(ab)^2}{2(c-b)} \tag{2}$$

$$b^3 + 3\,{}^{1}\!/_{10}\, b^2 = 1313783\,{}^{1}\!/_{10}$$

This has one real root,

$$b = 108\ {}^{1}/_{2}$$

whence

$$c = b + (c–b) = 114\ {}^{7}/_{10}$$

Problem 17 gives the quantities

$ac = 1337\ {}^{1}/_{20}$

$c–b = 1\ {}^{1}/_{10}$

and b is required. The cubic equation arrived at is

$$b^3 + \frac{5(c-b)}{2}b^2 + 2(c-b)^2 b = \frac{(ac)^2}{2(c-b)} - \frac{(c-b)^3}{2} \tag{3}$$

$$b^3 + 2\,{}^3\!/_4\, b^2 + 2\,{}^{21}\!/_{50}\, b = 812591\,{}^{59}\!/_{125}$$

This has one real root,

$b = 92\ {}^{2}/_{5}$

A comment explains (3) geometrically, but here our textual problems start, for a large number of characters are missing in the extant editions. In spite of the lacunae in the comment text it is clear that the geometrical construction is similar to that shown in Figures 3 and 4. The solid in Figure 3 has volume

$$\frac{c^2(c+b)}{2} - \frac{(c-b)^3}{2} = \frac{c^2\left(c^2-b^2\right)}{2(c-b)} - \frac{(c-b)^3}{2}$$

$$= \frac{a^2c^2}{2(c-b)} - \frac{(c-b)^3}{2}$$

which is the right side of (3). The sum of the volumes of the blocks into which the solid is divided (Figure 4) is

$$b^3 + 2(c-b)b^2 + \frac{(c-b)}{2}b^2 + (c-b)^2 b + 2\frac{(c-b)^2}{2}b$$

$$= b^3 + \frac{5(c-b)}{2}b^2 + 2(c-b)^2 b$$

which is the left side of (3).

Note that the dimensions of the triangles in Problems 15–17 are based on Pythagorean triples:

$$\begin{aligned} &15. \quad [14\,{}^{7}/_{20},\ 49\,{}^{1}/_{5},\ 51\,{}^{1}/_{4}] = [7, 24, 25] \times {}^{41}/_{20} \\ &16. \quad [37\,{}^{1}/_{5},\ 108\,{}^{1}/_{2},\ 114\,{}^{7}/_{10}] = [12, 35, 37] \times {}^{31}/_{10} \\ &17. \quad [14\,{}^{3}/_{10},\ 92\,{}^{2}/_{5},\ 93\,{}^{1}/_{2}] = [13, 84, 85] \times {}^{11}/_{10} \end{aligned} \tag{4}$$

Figure 3

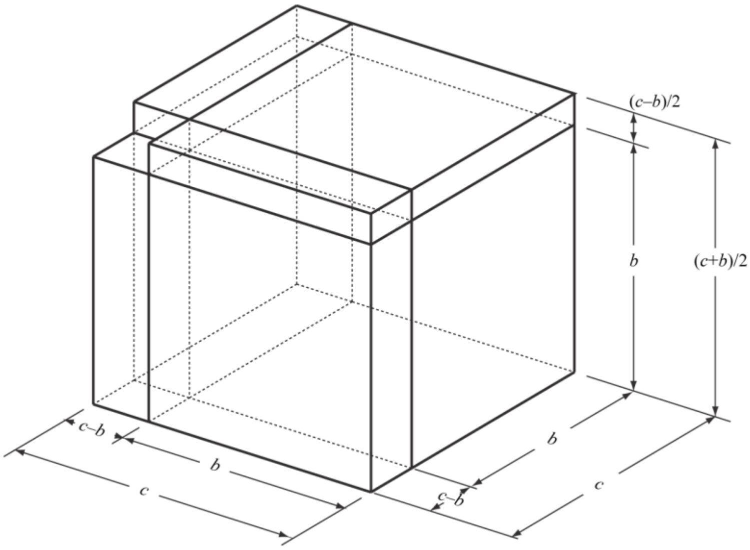
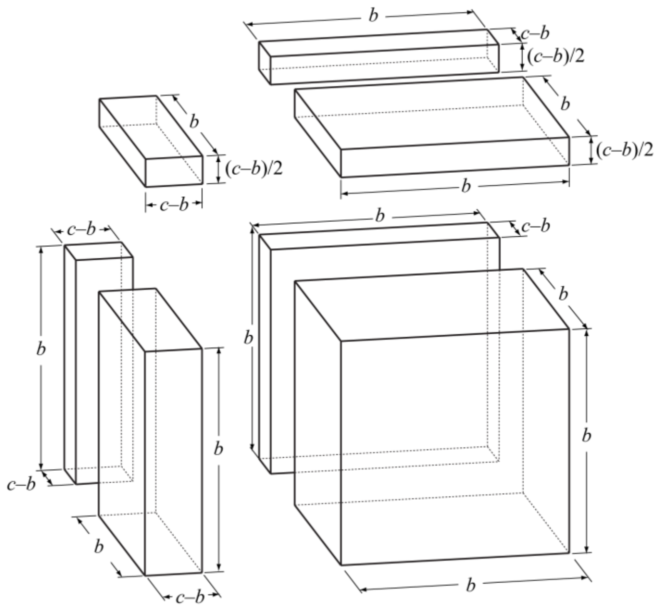


Figure 4



In **Problems 18–20** so much is missing from the text that any reconstruction will to some extent be speculative. The Chinese mathematician Zhang Dunren 張敦仁 (1754-1834) and the Korean mathematician Nam Pyŏng-Gil 南秉吉 (1820–1869) have given two reconstructions which agree in principle but differ in detail. Neither explains how he arrived at his reconstruction, but some matters seem clear.

Any reconstruction should satisfy the approximate number of characters seen to be missing from the text.[13] In addition, certain assumptions can safely be made: (1) The problems all concern right triangles; (2) the dimensions of the triangles are derived from Pythagorean triples, as in Problems 15–17; (3) the problem statements all follow the same rigid pattern as in Problems 15–17; and (4) the problems come in pairs, so that, just as 15 and 16 are equivalent, so are the pairs 17–18 and 19–20.

Assumption (4) implies that the fragmentary problem statements are
17. Given ac and $c–b$, determine b.
18. Given bc and $c–a$, [determine either a or b].
19. Given bc [and a], determine b.
20. Given [ac and] b, [determine a].

What remains of the 'method' for Problem 18 indicates that the quantity to be determined is b.

Zhang Dunren and Nam Pyŏng-Gil agree on the form of the problem statements as given here, but they differ on the numerical values of the given quantities in Problems 18 and 19. Their reconstructions are as follows:[14]

			Nam Pyŏng-Gil	Zhang Dunren
18	Given	bc =	$4428\ ^3/_5$	$4739\ ^3/_5$
		$c–a$ =	55	$54\ ^2/_5$
	Result	b =	66	68
	Other quantities	a =	$12\ ^1/_{10}$	$15\ ^3/_{10}$
		c =	$67\ ^1/_{10}$	$69\ ^7/_{10}$
19	Given	bc =	$^3/_{50}$	726

13 Assuming that the *Tianlu Linlang congshu* edition used the same page and line lengths as the Southern Song edition. This assumption is not certain, but the counts of missing characters may be taken as approximations.

14 In all but one case the two editors insert the same number of characters counted by the *Zhibuzuzhai* edition. In giving the value of bc in Problem 18, both insert 10 characters where the *Zhibuzuzhai* edition indicates only 9 missing.

		a =	$^{7}/_{100}$	$7\,^{7}/_{10}$
	Result	b =	$^{6}/_{25}$	$26\,^{2}/_{5}$
	Other quantity	c =	$^{1}/_{4}$	$27\,^{1}/_{2}$
20	Given	ac =	$164\,^{14}/_{25}$	
		b =	$16\,^{1}/_{2}$	
	Result	a =	$8\,^{4}/_{5}$	
	Other quantity	c =	$18\,^{7}/_{10}$	

The Pythagorean triples on which these reconstructions are based are:

18. $[12\,^{1}/_{10}, 66, 67\,^{1}/_{10}] = [11, 60, 61] \times {^{11}/_{10}}$
    $[15\,^{3}/_{10}, 68, 69\,^{7}/_{10}] = [9, 40, 41] \times {^{17}/_{10}}$
19. $[^{7}/_{100}, {^{6}/_{25}}, {^{1}/_{4}}] = [7, 24, 25] \times {^{1}/_{100}}$
    $[7\,^{7}/_{10}, 26\,^{2}/_{5}, 27\,^{1}/_{2}] = [7, 24, 25] \times {^{11}/_{10}}$
20. $[8\,^{4}/_{5}, 16\,^{1}/_{2}, 18\,^{7}/_{10}] = [8, 15, 17] \times {^{11}/_{10}}$

In the following we have arbitrarily chosen to follow Zhang Dunren's reconstructions of the given quantities. Zhang Dunren also attempts—with much less success—to reconstruct large parts of the 'method' sections, and we have not followed him in the translation. Nam Pyŏng-Gil's edition does not include Wang Xiaotong's methods, but instead gives his own, which appear to follow the methods of Li Ye 李冶 (1192–1279) in *Ceyuan haijing* 測圓海鏡.[15]

Problem 18 is, by assumption (4) above, equivalent to Problem 17. It results in the cubic equation

$$a^3 + \frac{5(c-a)}{2}a^2 + 2(c-a)^2 a = \frac{(bc)^2}{2(c-a)} - \frac{(c-a)^3}{2}$$

(cf. (3) above). In Zhang Dunren's reconstruction,

$$a^3 + 136a^2 + 5918\tfrac{18}{25}a = 125974\tfrac{233}{1000}$$

[15] See e.g. Mei Rongzhao 1966; Chemla 1982; Martzloff 1997: 143–149; Zhang Fukai 2005.

This has one real root,

$$a = 15\ {}^{3}/_{10}$$

And

$$b = \frac{bc}{(c-a)+a} = 68$$

Problems 19 and 20 are extremely fragmentary, but the assumption that they are equivalent makes Zhang Dunren's and Nam Pyŏng-Gil's reconstructions of the problems and their methods plausible. The method for Problem 19 leads in each case to the quadratic equation

$$x^2 + a^2 x = (bc)^2 \tag{5}$$

which in Zhang Dunren's version gives

$$x^2 + 59\ {}^{29}/_{100}\ x = 52706$$

where $x = b^2$. This has one positive root.

$$b = 26\ {}^{2}/_{5}$$

It is easy to derive (5) algebraically, for

$$b^4 + a^2 b^2 = b^2\left(a^2 + b^2\right) = (bc)^2$$

but what little remains of the comment appears to imply a geometric explanation. Commentators have had difficulty here, for a geometric explanation would seem to involve four-dimensional quantities. The lack of units in the quantities involved, however, makes the construction shown in Figure 5 plausible, in which both b and b^2 are used as linear measures. The volume of the solid is

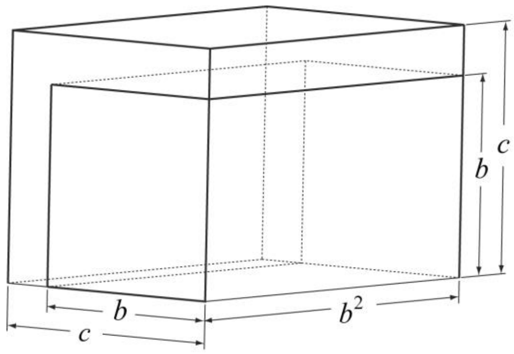
$$b^2c^2 = (bc)^2$$

which is the right side of (5), and the sum of the two blocks into which the solid is divided is

$$b^4 + \left(c^2 - b^2\right)b^2 = x^2 + a^2 x$$

which is the left side of (5). If our reconstruction is correct, the 'length' mentioned in one of the remaining fragments of the comment is probably x, the length of the solid.

Figure 5



Problem 20, assumed to be equivalent to Problem 19, results in the quadratic equation

$$x^2 + b^2 x = (ac)^2$$

where $x = a^2$. In Zhang Dunren's reconstruction,

$$x^2 + 272\,{}^1/_4\, x = 27079\,{}^{621}/_{625}$$

This has one positive root,

$$a = 8\,{}^4/_5$$

Concluding Remarks

How did Wang Xiaotong arrive at the results in his book? The question is probably unanswerable, but some guesses may be permissible.

We presume that he started with one or more general methods and derived a number of concrete geometric constructions which could be solved by their use. Our study of several of his problems in solid geometry[16] suggests that his most important method was dissections of 3-dimensional objects leading to (in modern terminology) cubic equations. (We hope that further research will lead to a more specific description.)

Our guess is then that Wang Xiaotong's study of the possibilities of this method led, in addition to the problems in solid geometry, to the three dissections related to right triangles reconstructed here.

Having found a relation between the dimensions of a right triangle and a cubic equation, Wang Xiaotong then needed only to fit a specific right triangle to it. In each case he used a Pythagorean triple, making the student's work more difficult by multiplying by a fraction. A general method for generating Pythagorean triples seems to have been known to Chinese mathematicians since the *Jiuzhang suanshu*,[17] but the three triples known to have been used by Wang Xiaotong (see (4) above) all have $c–b$ equal to 1 or 2. They could therefore be generated using the simpler special cases ascribed by Proclus to Pythagoras and Plato respectively:[18]

$$m^2 + \left(\frac{m^2-1}{2}\right)^2 = \left(\frac{m^2+1}{2}\right)^2$$

$$(2m)^2 + \left(m^2-1\right)^2 = \left(m^2+1\right)^2$$

We confidently expect that careful study of the solid-geometry problems in the book will reveal further examples of disguised Pythagorean triples, and that these will provide useful clues to Wang Xiaotong's general methods.

A further question is the function of this book in mathematical education. Though we know that it was adopted as one of the textbooks of Imperial mathematical education, we do not know whether it was originally written for this purpose, or perhaps was intended as a *tour de force* in pure mathematics. The main governmental uses of advanced mathematics—beyond accounting and the like—were in astronomy and in public works. It is difficult to see any use for these geometrical methods in astronomy, but in public works a thorough understanding of dissection methods (not necessarily the specific problems given in the book) might have been very

---

[16] These are Problems 2, 3, 4, 5, and 7, each of which consists of several related but distinct problems. Our interpretation of the four problems of Problem 2 is given in Lim and Wagner 2013.

[17] Chemla and Guo (2004: 727), under the interpretation of Van der Waerden (1983: 5–7).

[18] Heath 1921, 1: 80–81.

useful to an official calculating the volumes of earthworks and their labor requirements.

Chinese Text and Translation

In the Chinese text we indicate text variants with the notation {*a*/*b*/*c*}, where *a* is the 1684 version, *b* is Qian Baocong's version, which is usually the same as Zhang Dunren's, and *c*, if present, is from some other version, indicated in a footnote. We use Qian Baocong's punctuation.

Our translations of the smaller-character comments in the Chinese text are printed with a sideline. Our mathematical comments are given indented in the text, while a few philological comments are given in footnotes.

Problem 15

假令有句股相乘冪七百六 、五十分之一，弦多於句三十六、十分之九。問三事各多少？
答曰：
句十四、二十分之七，
股四十九、五分之一，
弦五十一、四分之一。
術曰：冪自乘，倍多數而一，為實。半多{　／數為}廉法，從。開立方除之，即句。以弦多{　／數加之}，即弦。以句除冪，即股。句股相乘冪自{　／乘，即}句冪乘股冪{　／之積。故}以倍句弦差而一，得一句與半差{　／相連，乘／再乘得[19]}句冪為方。故半差為廉{／法}，從，開立方除之。
[In a right triangle] the area obtained by multiplying the base [*a*] by the leg [*b*] is [*ab* =] 706 $^1/_{50}$ and the hypotenuse [*c*] is greater than the base [*a*] by [*c*–*a* =] 36 $^9/_{10}$. How large are the three quantities?
Answer:
base, [*a* =] 14 $^7/_{20}$
leg, [*b* =] 49 $^1/_5$
hypotenuse, [*c* =] 51 $^1/_4$
Method: Multiply the area [*ab*] by itself and divide by twice the difference [2(*c*–*a*)] to make the *shi* [the constant term]. Halve the difference [*c*–*a*] to make the *lianfa* [the quadratic coefficient]. Extract the cube root; this is the base [*a*].

[19] Zhang Dunren 1803.

$$a^3+\frac{(c-a)}{2}a^2=\frac{(ab)^2}{2(c-a)}$$

Add the hypotenuse difference [c–a]; this is the hypotenuse [c]. Divide the product [ab] by the base [a]; this is the leg [b].

$$c = a + (c - a)$$

$$b=\frac{ab}{a}$$

[Comment:] The product of the base and the leg, multiplied by itself [$(ab)^2$], is the 'volume' [*ji* 積] obtained by multiplying the area of [the square on] the base by the area of [the square on] the leg [a^2b^2].

$$(ab)^2 = a^2b^2$$

In the following see Figure 2.

Therefore dividing by the doubled difference between the base and the hypotenuse [$2(c–a)$] gives the base and the halved difference lined up [$a + (c–a)/2$] multiplied by the area of [the square on] the base [a^2] to make a box [*fang* 方, rectangular parallelepiped].

$$\frac{(ab)^2}{2(c-a)}=a^2\left(a+\frac{c-a}{2}\right)$$

This is why the difference is halved to make the *lianfa*, and the extraction of the cube root is carried out.

Problem 16

假令有句股相乘冪四千三十六、五分之{ ／一，股}少於弦六、五分之一。問弦多少？
答曰：弦一百一十四、十分之七。
術曰：冪自乘，倍少數而一，為實。半少為廉法，從。開立方除之，即股。加差，即弦。

[In a right triangle], the area obtained by multiplying the base [a] by the leg [b] is [ab=] 4036 $^1/_5$. The leg is less than the hypotenuse [c] by [c–b =] 6 $^1/_5$. How large is the hypotenuse?
Answer: The hypotenuse is [c=] 114 $^7/_{10}$.

This problem is equivalent to Problem 15.

Method: Multiply the area [*ab*] by itself and divide by twice the difference [*c*–*b*] to make the *shi* [the constant term]. Halve the difference [*c*–*b*] to make the *lianfa* [the quadratic coefficient]. Extract the cube root; this is the leg [*b*].

$$b^3 + \frac{c-b}{2} b^2 = \frac{(ab)^2}{2(c-b)}$$

Add the difference [*c*–*b*] to obtain the hypotenuse [*c*].

$$c = b + (c-b) = 114\ {}^{7}/_{10}$$

Problem 17

假令有句弦相乘冪一千三百三十七、二十分之一，弦多於股一、十分之一。問股多少？
答曰：九十二、五分之二。
術曰：冪自乘，倍多而一，為立冪。又多再{ ／自}乘，半之，減立冪，餘為實。又多數自乘，{ ／倍之， }為方法。又置多數，五之，二而一，為廉{ ／法，從}。開立方除之，即股。句弦相乘冪自{ ／乘，即句}冪乘弦冪之{ ／積。故以倍}股弦差而一，得一股與{半差 ／□□□□□□半差}為方今多再自乘半之為隅{ ／□□□□□□}橫虛二立廉{ ／□□□□□□□□□□□□□□}倍之為從隅{ ／□□□□□□□□□□□□□□}多為上廉即二多{ ／／□□□□□□□□□□□[20]}法故五之二而一{ ／□□□}
[In a right triangle] the area obtained by multiplying the base [*a*] by the hypotenuse [*c*] is [*ac* =] $1337^{1}/_{20}$, and the hypotenuse is greater than the leg by [*c*–*b* =] $1\ {}^{1}/_{10}$. How large is the leg [*b*]?
Answer: [*b*=] $92\ {}^{2}/_{5}$.
Method: Multiply the area [*ac*] by itself and divide by twice the difference [*c*–*b*] to make a volume $\left[V = \frac{(ac)^2}{2(c-b)}\right]$. Further multiply the difference [*c*–*b*] twice by itself, halve this, and subtract from the volume [*V*]. The difference is the *shi* [the constant term].
Further multiply the difference [*c*–*b*] by itself and double it to make the *fangfa* [the linear coefficient].
Further lay out the difference [*c*–*b*], multiply by 5, and divide by 2 to make the *lianfa* [the quadratic coefficient].

[20] Guo Shuchun & Liu Dun 1998, 2: 18.

Extract the cube root. This is the leg [*b*].

$$b^3+\frac{5(c-b)}{2}b^2+2(c-b)^2b=\frac{(ac)^2}{2(c-b)}-\frac{(c-b)^3}{2}$$

[Comment:] Multiplying the area [*ac*] obtained by multiplying the base by the hypotenuse by itself gives the 'volume' [*ji* 積] obtained by multiplying the area of [the square on] the base [a^2] by the area of [the square on] the hypotenuse [c^2].

$$(ac)^2 = a^2c^2$$

From here on the text is very fragmentary.

Therefore dividing by twice the difference [*c-b*] gives one leg [*b*] and. . .
[5 characters missing] . . . to make a box [*fang* 方]. The difference [*c–b*], multiplied twice by itself and halved, is the corner-piece [*yu* 隅] . . .
[5 characters missing] . . . two standing side-pieces [*li lian* 立廉], transverse [*heng* 橫] and empty [*xu* 虛] [?], . . .
[11 characters missing] . . . doubled to make the longitudinal corner-piece/s [*zong yu* 從隅], . . .
[11 characters missing] . . . the difference [*c–b*] makes the upper side-piece/s [*shang lian* 上廉] . . .
[9 characters missing] . . . *fa* 法 [coefficient?]. Therefore it is multiplied by 5 and divided by 2. . . .
[9 characters missing].

Problem 18

假令有股弦相乘冪{ ／四千七百三十九、五分之／四千四百二十八五分之}三，句少於弦五十{ ／四、五分之二。問股多少？／五問股多少}
答曰：六{ ／十八。／十六}
術曰：冪自乘，{ ／倍少數而一，為立冪。又少數}再自乘，半之，以{ ／減立冪，餘為實。又少數自}乘，倍之，為方法。{ ／又置少數，五之，二而一，為}廉法，從。開立方{ ／除之，即句。加差即弦。弦除}冪，即股。
[In a right triangle] the area obtained by multiplying the leg [*b*] by the hypotenuse [*c*] is [*bc* =] . . .
[9 characters missing] . . . 3. The base [*a*] is less than the hypotenuse [*c*] by [*c–a* =] 50 . . .
[characters missing] . . . Answer: 6 . . .
[characters missing] . . . Method: Multiply the area [*bc*] by itself . . .
[11 characters missing] . . . multiply twice by itself and halve it . . .
[10 characters missing] . . . multiply and double it to make the *fangfa* [the linear coefficient]. . . .

[10 characters missing] . . . *lianfa* [the quadratic coefficient]. Carry out the extraction of the cube root . . .
[10 characters missing] . . . the area gives the leg.

Problem 19

假令有股弦相乘冪{　/七百二十六，句七、十分之／五十分之三句一百分之}七。問股多少？
答曰：股二十{　/六、五分之二。／五分之六}
術曰：冪自{　／乘，為實。句自乘，為方法，從。開方}除之，所得，{　／又開方，即股}{／□□□□□□□□□□□□□□□□□□}數亦是股{　／□□□□□□□□□□□□□□□}為長以股{　／□□□□□□□□□□□□□□□}得股冪又開{　／。。。}股北分母常{　／。。。}

[In a right triangle] the area obtained by multiplying the leg [*b*] by the hypotenuse [*c*] is [*bc* =] . . . [10 characters missing] . . . 7. How large is the leg [*b*]?
Answer: The leg is 20 . . . [characters missing] . . . Method: The area . . . by itself . . . [12 characters missing] . . . divide it by . . . ; the result . . . [characters missing] . . .
[Comment:] [characters missing] . . . the number is also the leg [*b*] . . .
[characters missing] . . . to make the length [*chang* 長], . . . by the leg [*b*] . . .
[characters missing] . . . to obtain the area of [the square on] the leg [*b*]. Extract the . . . root . . .
[characters missing] . . . *gu bei fen mu chang* 股北分母常 [?]

Problem 20

假令有股十六、二分{　／之一，句弦相乘冪一百六}十四、二十五分{　／之十四。問句多少？}
答曰：{　／句八、五分之四。／八五分之四}
術曰：冪自乘{　／為實。股自乘，為方法，從。開方}除之，所得，又開方{　／即句。}
[In a right triangle] the leg [*b*] is 16 and [one] half . . . [10 characters missing] . . . 4 [and] . . . twenty-fifths . . . [characters missing] . . .
Answer: . . . [characters missing] . . .
Method: The area multiplied by itself . . . [11 characters missing] . . . extract the . . . root and extract the square root of the result . . . [characters missing]

References

Bao Tingbo 鮑廷博, ed. 1780. *Jigu suanjing* 緝古算經 (Continuation of ancient mathematics) (*Zhibuzuzhai congshu* 知不足齋叢書 edn). http://www.scribd.com/doc/64581316 (accessed 6 December 2013).

Bréard, Andrea. 1999. *Re-Kreation eines mathematischen Konzeptes im chinesischen Diskurs: 'Reihen' vom 1. bis 19. Jahrhundert*. Stuttgart: Franz Steiner Verlag (Boethius: Texte und Abhandlungen zur Geschichte der Mathematik und der Naturwissenschaften; 42).

Bréard, Andrea. 2002. "Problems of pursuit: Recreational mathematics or astronomy?" In *From China to Paris: 2000 years' transmission of mathematical ideas*, edited by Yvonne Dold-Samplonius et al. Stuttgart: Franz Steiner Verlag.

Chemla, Karine. 1982. *Étude du livre «Reflets des mesures du cercle sur la mer»*. Dissertation, University of Paris XIII.

———. 1994. "Similarities between Chinese and Arabic mathematical writings: (I) Root extraction." *Arabic Sciences and Philosophy* 4: 207–266. http://journals.cambridge.org/action/displayAbstract?aid=2201316 (accessed 6 December 2013).

Chemla, Karine, and Guo Shuchun. 2004. *Les neuf chapitres: Le classique mathématique de la Chine ancienne et ses commentaires*. Paris: Dunod.

Cullen, Christopher. 1993. "Chiu chang suan shu." In *Early Chinese texts: A bibliographical guide*, edited by Michael Loewe. Berkeley: Society for the Study of Early China and Institute of East Asian Studies, University of California, pp. 16–23.

Dai Zhen 戴震, ed. 1777. *Jigu suanjing* 緝古算經 (Continuation of ancient mathematics) (*Weibo Xie* 微波榭 edn). http://www.scribd.com/doc/46794333 (accessed 6 December 2013).

Guo Shuchun 郭书春 and Liu Dun 刘钝, eds. 1998. *Suanjing shishu* 算经十书 (Ten mathematical classics). 2 vols. Shenyang: Jiaoyu chubanshe. http://www.scribd.com/doc/53778139 (accessed 6 December 2013).

He Shaogeng 何绍庚. 1989. "«Jigu suanjing» gougu ti yiwen shi bu" «缉古算经»勾股题佚文试补 (Attempted reconstruction of the fragmentary text on right triangles in *Jigu suanjing*). *Zhongguo lishi bowuguan guankan* 中国历史博物馆馆刊 (Journal of the Museum of Chinese History) 12: 37–43, 54. http://www.scribd.com/doc/189844342 (accessed 6 December 2013).

Heath, Thomas. 1921. *A history of Greek mathematics*. 2 vols. Oxford: Clarendon Press.

Horner, W. G. 1819. "A new method of solving numerical equations of all orders, by continuous approximation." *Philosophical Transactions of the Royal Society of London* 109: 308–335. http://www.jstor.org/stable/107508 (accessed 6 December 2013).

Horng, Wann-Sheng 洪萬生. 2002. "Sino-Korean transmission of mathematical texts in the 19th century: A case study of Nam Pyŏng-gil's *Kugo Sulyo Tohae* 勾股述要圖解." *Historia Scientiarum* 12: 87–99. Not seen. http://cat.inist.fr/?aModele=afficheN&cpsidt=14716065 (accessed 6 December 2013).

Huang Juncai 黃俊才 (Chun-Tsai Huang). 2008. "Li Huang «Jigu suanjing kaozhu» zhi neirong fenxi" 李潢《緝古算經考注》之內容分析 (Analysis of the content of Li Huang's edition of *Jigu suanjing*). M.A. thesis, National Taiwan Normal University.

Jiu Tang shu 1975. 舊唐書 (Older standard history of the Tang Dynasty; AD 618–907, critical edn). Shanghai: Zhonghua shuju.

Lam Lay-Yong [Lan Lirong 蓝丽蓉]. 1970. "The geometrical basis of the ancient Chinese square-root method." *Isis* 61.1: 92–102. http://www.jstor.org/stable/229151 (accessed 6 December 2013).

———. 1977. *A critical study of the Yang Hui Suan Fa, a 13th-century Chinese mathematical treatise*. Singapore: Singapore University Press.

———. 1986. "The development of polynomial equations in traditional China." *Mathematical Medley* (Singapore Mathematical Society) 14.1: 9–34.

Li Huang 李潢, ed. [ca. 1832]. *Jigu suanjing kaozhu* 緝古算經考注. Jing Yushu 靖玉樹, ed., *Lidai suanxue jicheng* 歷代算學集成, 1994. http://www.scribd.com/doc/43099474 (accessed 6 December 2013).

Libbrecht, Ulrich. 1973. *Chinese mathematics in the thirteenth century: The Shu-shu chiu-chang of Ch'in Chiu-shao*. Cambridge, Mass. & London: M.I.T. Press (M.I.T. East Asian science series; 1).

Lin Yanquan 林炎全. 2001. *Jigu suanjing tantao* 輯古算經探討 (A study of *Jigu suanjing*). Taiwansheng zhongdeng jiaoshi yanxi hui 台灣省中等教師研習會. http://www.scribd.com/doc/78851923 (accessed 6 December 2013).

Lim, Tina Su-lyn, and Donald B. Wagner. "The Grand Astrologer's platform and ramp: Four problems in solid geometry from Wang Xiaotong's 'Continuation of ancient mathematics' (7th century AD)", *Historia Mathematica*, 2013, 40.1: 3–35. http://www.sciencedirect.com/science/article/pii/S0315086012000596 (accessed 6 December 2013).

Martzloff, Jean-Claude. 1997. *A history of Chinese mathematics*. Translated by Stephen S. Wilson. Berlin/Heidelberg: Springer-Verlag. Corrected second printing, 2006. Orig. *Histoire des mathémathiques chinoises*, Paris: Masson, 1987.

Mei Rongzhao 梅荣照. 1966. "Li Ye ji qi shuxue zhuzuo" 李冶及其数学著作 (Li Ye and his mathematical works). In *Song Yuan shuxue shi lunwenji* 宋元数学史论文集 (Essays on the history of mathematics in the Song and Yuan periods, AD 960–1368), ed. by Qian Baocong 錢寶琮. Beijing: Kexue chubanshe, pp. 104–148.

Nam Pyŏng-Gil 南秉吉. (19th cent.) *Chipgo yŏngdan* 緝古演段 (Edition of *Jigu suanjing*) (韓國科學技術史資料大系 [Collected materials for the history of science and technology in Korea]). Facs. repr. 1985. http://www.scribd.com/doc/27920990 (accessed 6 December 2013).

Qian Baocong 錢寶琮, ed. 1963. *Suanjing shi shu* 算經十書 (Ten mathematical classics). Beijing: Zhonghua Shuju. http://www.scribd.com/doc/53797787 (accessed 6 December 2013).

———. 1966. "Wang Xiaotong «Jigu suanjing» di er ti, di san ti shuwen shuzheng" 王孝通«缉古算经»第二题,第三题术问疏証 (Explication of problems 2 and 3 in Wang Xiaotong's *Jigu suanjing*). *Kexueshi jikan* 科学史集刊 (Science History Periodical) 9: 31–52. http://www.scribd.com/doc/189821410 (accessed 6 December 2013).

———. 1998. "Suanjing shishu" 算经十书 (Ten mathematical classics). In *Li Yan Qian Baocong kexueshi quanji* 李俨钱宝琮科学史全集 (Collected works of Li Yan and Qian Baocong on the history of science). Shenyang: Liaoning jiaoyu chubanshe, vol. 4.

Rees, Paul K., and Fred W. Sparks. 1967. *College algebra*. 5th ed. New York: McGraw-Hill.

Shen Kangshen 沈康身, John N. Crossley, and Anthony W.-C. Lun. 1999. *The nine chapters on the mathematical art: Companion and commentary*. Oxford / Beijing: Oxford University Press / Science Press.

Van der Waerden, B. L. 1983. *Geometry and algebra in ancient civilizations*. Berlin/Heidelberg/New York/Tokyo: Springer-Verlag.

Wang Ling 王鈴, and Joseph Needham. 1955. "Horner's method in Chinese mathematics: Its origins in the root-extraction procedures of the Han dynasty." *T'oung Pao* 43.5: 345–401. http://www.jstor.org/stable/4527405 (accessed 6 December 2013).

Wang Rongbin 王荣彬. 1990. "Wang Xiaotong «Jigu suanjing» zizhu yiwen jiaobu" 王孝通«缉古算经»自注佚文校补 (Reconstruction of the missing text in Wang Xiaotong's commentary to his *Jigu suanjing*). In *Shuxue shi yanjiu wenji, di yi ji* 数学史研究文集，第一辑 (Studies in the history of mathematics, 1), edited by Li Di 李迪. Hohhot/Taibei: Neimenggu daxue chubanshe/Jiuzhang chubanshe, pp. 50–55.

Ying, Jia-ming 英家銘. 2011. "The *Kujang sulhae* 九章術解: Nam Pyŏng-Gil's reinterpretation of the mathematical methods of the *Jiuzhang suanshu*." *Historia Mathematica* 38.1: 1–27. http://www.sciencedirect.com/science/article/pii/S0315086010000273 (accessed 6 December 2013).

Zhang Dunren 張敦仁, ed. 1803. *Jigu suanjing* 緝古算經 (Continuation of ancient mathematics) (*Congshu jicheng* 叢書集成). Colophon dated 1803. http://www.scribd.com/doc/48275688/ (accessed 6 December 2013).

Zhang Fukai (Fu-kai Chang) 張復凱. 2005. "Cong Nan Bingji (1820–1869) «Yigu yanduan» kan dong suanshi shang tianyuan shu yu jiegen fang zhi 'duihua'" 從南秉吉(1820~1869)《緝古演段》看東算史上天元術與借根方之「對話」 (On the dialogue between the Chinese *Tianyuan shu* and the Western *Jiegen fang*: A Korean version by Nam Pyŏng-Gil (1820-1869)). M.A. Thesis, National Taiwan Normal University. http://ndltd.ncl.edu.tw/cgi-bin/gs32/gsweb.cgi?o=dnclcdr&s=id=%22093NTNU5479004%22.&searchmode=basic (accessed 6 December 2013).