Numerical Tables in Chinese Writings Devoted to Mathematics: From Early Imperial Manuscripts to Printed Song-Yuan Books

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** Abstract:** This article establishes that the discursive parts of the earliest known mathematical manuscripts in Chinese were composed of (at least) two types of elements, marked by two types of texts. The manuscripts alternate continuous text, and text for numerical tables (what I call table-relations). I show that in these manuscripts, the latter were written down as ‘textual tables,’ and that two basic types of style were used for these textual tables. By contrast, tabular layouts have been used for a Qin period object and a Dunhuang manuscript carrying numerical tables. I suggest that these...
artifacts should be interpreted as computing tools. I further argue that, at least from the eleventh century onwards, diagrammatic tables were introduced into mathematical writings. They were used to write down new types of numerical tables. Diagrammatic features of such texts, like horizontal, vertical and oblique lines, played a key part in the reading, interpretation and use of these table-relations. In this sense, they can be compared with the Qin computing tool. I conclude that the fact that in Song-Yuan times these diagrammatic tables are referred to as ‘diagram 甲’ curiously echoes with the history of visual tools attested to in relation to mathematical activity in China.

This article deals with numerical tables attested to in mathematical writings composed in China up until the Song and Yuan dynasties. However, it concentrates more specifically on the numerical tables found in the earliest known manuscripts produced in early imperial China and devoted to mathematics.

The corpus of sources on which I focus is thus relatively clear, though I will be more explicit about it in what follows. Let me now clarify which textual phenomena in these documents will be at the heart of my analysis. The reason why clarification is required is simple. In general, when using the English term ‘table,’ one can be referring to two distinct kinds of realities. We need to make these two meanings explicit, not only to spell out the topic of this article, but also because the distinction is essential for my purposes.

On the one hand, the term ‘table’ in the expression ‘numerical table’ can be used to designate a type of content in a document, as well as features of its textual implementation. In this case, the term refers to the fact that the piece of writing designated as a ‘table’ expresses quantitative relationships in a specific way. The prime example for this is the so-called ‘multiplication table.’ The quantitative relationship expressed by such a table is that of multiplication. The range of numbers it covers varies from table to table. The specific feature of the texts sharing the label ‘multiplication table’ that justifies the use of the term ‘table’ is that they are all composed as a list of similar clauses, each of which makes explicit the result obtained by multiplying two specific numbers. More generally, we use the expression ‘numerical table’ when quantitative relationships are likewise expressed by means of a list of clauses, which each bring numerical values in relation to each other.²

² In this article, I use in general “writing” to refer to what is written in a document, while I keep the term “text” to designate how it is written, that is, how a writing is realized in its written form.

³ Sometimes, a table expresses not only a single quantitative relationship, but in fact a set of such relationships. For instance, the two relationships of doubling and tripling can be expressed in the form of a single numerical table, by means of the following sequence of clauses: “(1, 2, 3), (2, 4, 6), (3, 6, 9), (4, 8, 12), (5, 10, 15) …” In
On the other hand, the term ‘table’ also refers to a specific type of layout, such as the one which is included in the commentary *Detailed Explanations of The Nine Chapters on Mathematical Methods*, completed by Yang Hui 楊輝 in 1261 (see Figure 1.)\(^4\) Naturally, the two meanings of the term can be connected. Figure 1 shows an example of a numerical table the text of which has a tabular layout—we will return to it. But this is not always the case.

In order to distinguish between the two meanings of the term ‘table,’ when the context does not suffice to make clear which meaning I am using, I introduce a specific terminology. I will refer to the former use of ‘table’ by means of the expression ‘table-relation,’ whereas I will designate the latter as ‘table-layout.’ Using these terms, I can formulate the main issue addressed in this article as follows: which types of textual inscriptions have practitioners of mathematics shaped, or used, in ancient China to write down ‘table-relations’? I will argue that the question is important, because the evidence available at the present time seems to indicate that, between the earliest known mathematical writings and those produced in the Song-Yuan time period, there was a major shift in the types of text used to formulate ‘table-relations.’ The earliest extant mathematical manuscripts bear witness to the shaping, or the use, of specific types of text for ‘table-relations.’ I will designate these types of text by the expression ‘textual tables.’ By contrast, the Song-Yuan mathematical books testify to the use of new types of text for ‘table-relations,’ which I will call ‘diagrammatic tables.’

Note that so far, I have used the term ‘table’ as an observer’s category. We will further have to inquire into whether we can find actors’ categories for these objects, and also for the texts by means of which these objects were written down.

To formulate my aim more precisely, the main purpose of the article is, first, to describe the various types of text used in Chinese mathematical writings to write down ‘table-relations,’ and, second, to establish the shift evoked above. The first task will require that emphasis be placed mainly on

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\(^4\) It is not clear whether this part of the text was actually written by Yang Hui himself or whether it belongs to the c. eleventh century commentary by Jia Xian 賈憲. I believe that it belongs to the latter layer, which is the third out of four layers of commentaries and sub-commentaries on the Han canon *The Nine Chapters* gathered in Yang Hui’s book. However, I will present my arguments in another publication. On the question of the separation between the two layers of commentary mentioned here, compare Guo Shuchun 郭書春 (1988). I return to *The Nine Chapters* below.
the first time period, before the Song-Yuan dynasty, and mainly on manuscript evidence. I will show that manuscripts testify to the existence of two types of text for textual tables. The identification of these types will enable us to examine which quantitative relations were expressed by means of ‘textual tables’ and to also grasp transformations in this respect.

Types of Text used for ‘Table-Relations’ in Qin and Han Mathematical Manuscripts

To begin with, let us concentrate on the mathematical manuscripts from early imperial China recently excavated or bought on the antiquities market. They will provide evidence for our analysis of practitioners’ choices of textual format to write down ‘table-relations’ in early imperial China.

The first mathematical manuscript that came to light in an excavation was found during the winter 1983-1984 in a tomb sealed c. 186 BC at Zhangjiashan 張家山 (Jingzhou 荊州, Hubei Province 湖北省). It bore the title Suanshu shu 算術書 (Writings on Mathematical Procedures)5 Peng Hao 彭浩 (2001), pp. 4-6, notes that the content of its various sections does not seem to date from the same time period. In his view, the writing borne by the manuscript displays a set of features that echoes the administrative duties of a local government official during the Qin dynasty. At the same time, other features of the document rather reflect early Han dynasty regulations.

A few years after the publication of the Suanshu shu, several new manuscripts were found. For instance, a manuscript entitled Shu 數 (Mathematics) was bought on the Hong Kong antiquities market and is now kept at the Yuelu Academy in Changsha.6 The editors suggest that the writing borne by the manuscript was completed no later than 212 BC, and seems to reflect the

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6 This manuscript has been the topic of a PhD thesis: Xiao Can 肖燦 (2010), now published as Xiao Can 肖燦 (2015). A reproduction of the slips, and also a transcription and critical edition of Mathematics are published in Zhu Hanmin 朱漢民 and Chen Songchang 陳松長 (gen. eds.) (2011).
social practice of the Qin kingdom or the Qin dynasty. In other words, it was probably completed decades before the Suanshu shu was. Moreover, a manuscript entitled Suanshu 算術 (Mathematical Procedures) and probably dated from before c. 157 BC has been found in tomb M77 at Shuihudi 睡虎地 (Yunneng 雲夢, Hubei 湖北 Province). A general description and the photo of ten of its slips were published. According to the first report about the tomb, the manuscript has features in common as well as differences with the Suanshu shu. However, we must wait for the publication of the entire document to assess the validity of this assertion.

**A First Type of Text for a ‘Table-Relation’**

The Chinese text of seven clearly consecutive slips of Suanshu 算術 (Mathematical Procedures) is reproduced in Figure 2 in a way that is faithful to the original layout. As can be seen, the layout consists of four superposed parts (or ‘registers’). The upper register is to be read first, each piece of vertical text being followed by the one immediately to its left that begins at the same level. One then turns to the portion of the text placed in the register immediately under the first one and reads it in the same way. The subsequent registers of the text are read accordingly. That the text on these slips must be read in this way is clearly demonstrated by the fact that the text in question is that of a ‘table-relation,’ and that the order of the clauses makes clear which clause follows which. Let us explain why. The text of the upper two registers can be translated as follows:

(a) One multiplied by one, one.
(b) One multiplied by ten, ten.

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8 Hubei sheng wenwu kaogu yanjiusuo 湖北省文物考古研究所 and Yunneng xian bowuguan 雲夢縣博物館 (2008) provides a report of the excavation and photos of slips. Chemla and Ma (2011) publish a transcription and a tentative interpretation of seven mathematical slips in the published photo.
9 The translation and the edition of this passage are taken from Chemla and Ma (2011), pp. 168-173. The reader can find there the translation of the complete text written down on the seven slips and a discussion on its interpretation. The translation of the portion of text placed in one of the registers is separated from that of the next register by an empty line. The empty line corresponds to the fact that the upper register is separated from the second register following it by an empty space extending across all the strips. The same holds true for the subsequent registers. Each of the portions of text placed in a vertical segment of one of the registers is translated in a separate paragraph. Moreover, like in the other publication, I added letters between brackets at the beginning of the lines to make reference to them easier.
(c) One multiplied by a hundred, a hundred.
(d) One multiplied by a thousand, a thousand.
(e) One multiplied by ten thousand, ten thousand.
(f) Ten multiplied by ten, a hundred.
(g) Ten multiplied by a hundred, a thousand.
(h) Ten multiplied by a thousand, ten thousand.
(i) Ten multiplied by ten thousand, a hundred thousand.
(j) A hundred multiplied by a hundred, a hundred thousand.\(^{10}\)
(k) A hundred multiplied by ten thousand, a million.
(l) A thousand multiplied by ten thousand, ten million.\(^{11}\)
(m) Ten thousand multiplied by ten thousand, a hundred million.
(n) A hundred multiplied by a hundred, ten thousand.\(^{12}\)

The registers placed under these upper two registers deal with other topics. We thus have here a part of a document that presents a thematic unity. The quantitative relation that the table expresses is that of multiplication. However, by contrast to the common present-day ‘multiplication table’—for which there exist Chinese versions from early imperial China that I will evoke below—, that table-relation lists clauses that express which power of ten is obtained through the multiplication of two powers of ten. Let us call

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\(^{10}\) As was explained in Chemla and Ma (2011), p. 272, fn. 20, this clause and the previous one share the same result. However, if we rely on the pattern established in sentences (a) and (f), one may surmise that the text of the ‘table-relation’ originally had, between these two sentences, the following sentence: 百乘百，萬也 “A hundred multiplied by a hundred, ten thousand.” Note that sentence (m) also has a pattern of this type. It seems that the scribe forgot this sentence while writing down the text, and that, realizing his omission, he added the sentence at the end of the table, as sentence (n). However, one may also note that an omission of clauses of this type occurs twice, since there is no sentence 千乘千，百萬也 “A thousand multiplied by a thousand, a million,” corresponding to the pattern described above. It may thus very well be that the scribe deleted two sentences of the same type.

\(^{11}\) See footnote 10. The largest result in the table is qian wan “one thousand ten-thousand.” Going one step further would require introducing a new terminology.

\(^{12}\) See footnote 10.
such table-relations ‘multiplication tables for powers of ten,’ and refer to the table above as ‘Table 1.’

The text of each clause has the same structure, which can be represented by the following pattern: “a power of ten multiplied by a second power of ten, a third power of ten ye 也.” We note that the text of each clause is placed in a vertical slot of a register. Its conclusion is marked by the use of the same final particle, ye, as well as by an empty space on the slip. A clause thus corresponds, from a syntactical viewpoint, to a specific type of statement, and, from a material viewpoint, to a specific spatial inscription. Note that the other slips from Suanshu, the photo of which was published in Hubei sheng wenwu kaogu yanjiusuo 湖北省文物考古研究所 and Yunmeng xian bowuguan 雲夢縣博物館 (2008), are written in continuous text. The presence of a table is thus signaled by the specific material features of its textual inscription. We will return to this point.

The set of clauses composing the table presents a specific orientation in two ways.

To begin with, the first power of ten (first operand of the multiplication) is always smaller or equal to the second (second operand). This feature is correlated to the fact that there is no repetition, in the sense that the reader would be once given a clause regarding the multiplication of $x$ by $y$ and then a second one about the multiplication of $y$ by $x$. The property of the multiplication that it yields the same result in the two cases has been exploited to insert about only half the set of clauses in the table. The operands of the remaining clauses have been ordered accordingly.

Moreover, the clauses are organized according to an order of increasing powers of ten. The first five clauses have ‘one’ as their first operand and their second operand increases from ‘one’ to ‘ten thousand.’ Once this cycle is completed, the first operand becomes ‘ten’ whereas the second operand takes, in the following clauses, all values between ‘ten’ and ‘ten thousand,’ and so on. Despite exceptions, indicated in footnotes to the translation, the text of the table presents a clear and regular organization of the set of clauses. For tables organized in similar ways, the first operand in a cycle of clauses can be omitted after the first clause of the cycle (this is, for example, the case for the table of Suanshu shu given for the computation of volumes, which is mentioned below).

This organization of the set of clauses makes it clear that the reading of the table-relation must be carried out one register after the other, from top to bottom. While here the content imposes a mode of reading of the text, conversely the mode of reading reveals that the table-relation has been textually inscribed according to a specific material format. The format is characterized by two main features. It presents registers and supposes that the reader ‘circulates’ in a specific way between the portions of text inscribed in the slots of the registers. Moreover, each clause is inscribed in a vertical slot.
of a register, each of these slots containing a full clause and only a clause. The text of a clause is thus marked out in a material way.

As has been pointed out by Liu Jinhua (2003), this format has been regularly chosen to write down tables in Chinese manuscripts. This holds true for manuscripts written on slips as well as on wooden boards. For instance, the multiplication table between numbers ranging from 1 to 9 (and occasionally \( \frac{1}{2} \)) that is inscribed on the Qin wooden tablet J1 (6) 1 unearthed at Liye 里耶 is written down with the same type of text. It displays thirty-eight clauses from ‘nine times nine, eighty-one’ till ‘twice a half, one’ in six registers from top to bottom, separated from each other by a space extending horizontally, across the surface of the wooden tablet. Owing to the organization of the clauses recorded in J1 (6) 1, which is identical to the one described previously, we know that the text requires the same mode of reading of the registers one after the other. In other words, the content of the table is distinct from the one above, but the layout chosen for writing down the ‘table-relation’ as well as the reading of the layout are the same.

Each slot of each register contains a clause of the ‘table-relation’ (except the very last slot). In relation to the fact that the table is written on a tablet and not on slips, the number of clauses in the different registers varies.

13 In Chemla and Li (Forthcoming), Li Liang has included examples of tables found in early imperial manuscripts dealing with astral sciences that have the same layout. Since these manuscripts were excavated in different geographical locations, this shows the widespread use of such a layout to write down tables.

14 See the transcription and the black and white reproduction of the document in Zhang Chunlong 张春龙 and Long Jingsha 龙京沙 (2003), pp. 8, 9, respectively. A critical edition and a discussion of the wooden tablet are given in Wang Huanlin 王焕林 (2007), pp. 25-27, 176-190. A colour photograph is provided in plate III. The discussion includes reference to all similar tables found in early imperial manuscripts (p. 177). Since the latter book was published, however, a new wooden board has been found that bears exactly the same table. It is the wooden board M-025, which is kept among Qin documents at Peking University (Han Wei 韩巍 (2012), p. 3). The same Qin documents contain several sets of bamboo slips on mathematics, and Part A of Suanshu 算書 (Mathematical Writings), written in the fourth set of slips, has a similar table (Han Wei (2012), p. 3). I refer to a ‘table-relation’ of that kind using the expression ‘multiplication table for digits.’

15 Note that in the publications about J1 (6) 1 mentioned in the previous footnote, the transcriptions of the text of the table present a layout for the table different from the one borne by the actual wooden tablet. The clauses put on the tablet in the succeeding slots of a given register are transcribed horizontally, following each other in the same paragraph, and separated from each other by an empty space. As a consequence, the horizontal alignments of the beginning of the clauses placed in the same register are not reproduced.

16 There are six clauses in the first register, seven in the following two, eight clauses in the fourth register, seven in the fifth and three in the sixth. The three
Accordingly, the clauses are not as neatly organized from a material viewpoint as in the ‘multiplication table for powers of ten’ discussed above. The way of writing the clauses is also identical to the one described previously, the first operand being always smaller or equal to the second. Finally, the same property of the operation of multiplication is used to reduce by about a half the amount of clauses inserted in the document.

The table-relations inserted in the manuscript *Mathematics (Shu 數)* are all textualized in the same way, *except for one*, as we explain below. However, this manuscript also suggests that some nuances should be taken into consideration.

**What is a Clause?**

To explain this point, let us examine an example. Xiao Can describes the following table-relation (Table 2) as written in four registers (Xiao Can 月燦

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clauses in the sixth register leave a large space under them. Starting as a fourth clause, and extending until the lower edge of the tablet, one finds a concluding sentence, which provides the sum of the results of all clauses on the tablet. This feature does not occur in the ‘multiplication table for powers of ten’ discussed. However, many documents attest to similar ending. We return to this detail below. The first six items of the various registers are, roughly speaking, aligned vertically. These vertical alignments are more or less indicated in the layout of the transcriptions, through a vertical alignment of the first characters of the clauses from one paragraph to the next, for instance in Wang Huanlin 王煥林 (2007). However, the empty space caused in the first register by the inclusion of only six clauses is not indicated. The extra clause inserted to the left of the tablet in the fourth register does not appear as such in the layout of the transcriptions either. According to the description of the Qin wooden board M-025 given in Han Wei 韓巍 (2012), p. 3, we know that the inscription of the same table, including the identical sum of all results (1113) at the end, uses registers in a similar, but slightly different way, with respect precisely to the arrangement of the columns. Perhaps is it written more regularly. The table from the same group of Qin documents kept at Beida and written on bamboo slips occupies 8 slips, with 5 registers, and apparently presents differences with the previous tables (Han Wei 韓巍 (2012), p. 3). We return below to this table.

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17 Wang Huanlin 王煥林 (2007), pp. 183-184, gathers occurrences of the same ‘table-relation’ in the documents handed down from Chinese antiquity. See also Guo Shirong 郭世榮 (1998), pp. 358-359. All clauses share the same feature. Note that in the ‘multiplication table for digits’ the first operand describes shorter cycles than the second operand. The reverse holds true in the ‘multiplication table for powers of ten’ described. In correlation with this, the second operand decreases in the ‘multiplication table for digits’, while it increases in Table 1.

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18 Xiao Can 月燦 (2010), p. 10; id. (2015), p. 13, formulates the hypothesis that the pieces of text in *Mathematics* that are written in register formats could be “numerical tables.”
(2010), p. 51, fns 2, 1, 1, resp.; id. (2015), p. 64, idem). Let us quote it, slip by slip:

□¹⁹乘三分丿，二參而六_=，（六）分一也；²⁰
半乘半，四分一也；
/mark of chord/ 四分乘四分，四_=（四）十²¹六_=，（十六）
分一也；
少半乘一，少半也。/0410/

三分乘四分丿，三十四=二_=，（十二）分一也；
三分乘三分，三_=（三）而九_=，（九）分一也；²²
/mark of chord/ 少半乘十，三有（又）少半也；
五分乘六分，五六卅=，（卅）分之一也。/0778/

五分乘五分，五_=（五）廿=五_=，（廿五）分一也。/mark of chord/ /space/0774/

(1.1) [A half] multiplied by a third²³丿，two times three, six,

²⁹ Xiao Can 肖燁 (2010), p. 51; id. (2015), p. 64, and Zhu Hanmin 朱漢民 and Chen Songchang 陳松長 (2011), p. 74, fn 2, both note that the head of the slip is broken and that the meaning allows one to assume that the character 一半 ban ‘one half’ was written at the beginning of the slip.

²⁰ Xiao Can 肖燁 (2010), p. 51, fn2, resp.; id. (2015), p. 64, idem, and Zhu Hanmin 朱漢民 and Chen Songchang 陳松長 (2011), p. 74, fn 4, note that the space between this clause and the following one is not clear.

²¹ Zhu Hanmin 朱漢民 and Chen Songchang 陳松長 (2011), p. 74, fn 3, remarks that the repetition sign has been omitted (see parallel sign on slip 0778, clause one) and must be restored.

²² Xiao Can 肖燁 (2010), p. 51, fn1; id. (2015), p. 64, idem, and Zhu Hanmin 朱漢民 and Chen Songchang 陳松長 (2011), p. 75, fn 1, note that the space between clause 2 and clause 3 is not manifest. However, in this case the chord seems to have marked the beginning of the lower register.

²³ Anicotte (2012 (29 September)) has studied the expression of fractions in the Suanshu shu, especially the divide between the two forms of expression ‘n parts n fen n 分’ and ‘one of n parts n fen yi n 分之一’ (sometimes as ‘one of n parts n fen zhi yi n 分之一’ in ‘one thirtieth’) to designate the unit fraction 1/n. Like in the Suanshu shu (see below), in this table the operands are expressed in the form ‘n parts n fen n 分,’
one sixth;

(1.2) a half multiplied by a half, one fourth;\textsuperscript{24}

(1.3) a fourth multiplied by a fourth, four times four, sixteen, one sixteenth;

(1.4) a lesser half multiplied by one, a lesser half. \textit{slip 0410/}

(2.1) A third multiplied by a fourth. three times four, twelve, one twelfth;

(2.2) a third multiplied by a third, three times three, nine, one ninth;

(2.3) a lesser half multiplied by ten, three and a lesser half;

(2.4) a fifth multiplied by a sixth, five times six, thirty, one thirtieth. \textit{slip 0778/}

(3.1) A fifth multiplied by a fifth, five times five, twenty-five, one twenty-fifth. \textsuperscript{25}(3.2) A fourth multiplied by a fourth, four times four, sixteen, one sixteenth. \textit{slip 0774/}\textsuperscript{26}

Clearly, this is a ‘table-relation,’ for yet another quantitative relationship of multiplication, and at first sight, it seems to be textualized in the same way, using registers. Note that by contrast with the translation of Table 1 from \textit{Suanshu} above, in the translation of Table 2, an empty line represents a change of slip. In fact, in the case of Table 2, the empty spaces running across

whereas the result is expressed as ‘one of \textit{n} parts \textit{n} \textit{fen} \textit{yi} \textit{分}–.’ I translate the difference by distinguishing between ‘a \textit{n}-th’ and ‘one \textit{n}-th.’ These remarks are not valid for specific expressions for fractions such as ‘\textit{ban} \frac{1}{2} a half’ or ‘\textit{shuoban} 少半 a lesser half.’ Note that in this clause and others, a third is expressed as ‘\textit{san} \textit{fen} 三分 three parts,’ whereas in the last clause of the register, it is expressed as ‘a lesser half.’ In fact, there is a correlation between the use of the latter expression and the fact that the other operand of the multiplication is an integer. Also note the use of a punctuation sign after the first segment of the clause. The same holds true for the sentence occurring in the same register in the slip that follows.

\textsuperscript{24}Note that in contrast to similar clauses, this one does not include the related statement of the ‘multiplication table for digits.’

\textsuperscript{25}Note that by contrast to the other slips on which the table-relation is inscribed, there is no space here, but a punctuation mark.

\textsuperscript{26}The critical edition and explanations of the text can be found in Xiao Can 肖燦 (2010), p. 51; id. (2015), p. 64, as well as in Zhu Hanmin 朱漢民 and Chen Songchang 陳松長 (2011), pp. 74-75. The reproduction of the slips is on p. 12.
the three slips, and thereby marking registers, are not clearly marked on all slips (this holds true in particular for the distinction between registers 1 and 2). It further appears that, as for Table 1 (see Figure 2), the middle chord linking the slips marks the beginning of a lower register. For each slip, I represented both types of marks (even in cases when they are unclear) by line feeds. To refer to segments of the text more conveniently, I have introduced numbers to the left of the translation of the clauses that seem to be distinguished, if we assume that, like above, the text is inscribed within four independent registers \( n.m \) thus indicates that the clause is in register \( m \) of slip \( n \). Indeed, it seems that in a way similar to Table 1, which is reproduced in Figure 2, the text of each clause is placed in a vertical slot of a register, and its conclusion is marked by the use of the same final particle \( ye \) as well as by an empty space on the slip, or by the middle chord. Moreover, the size of the space on the slips seems to have been chosen in such a way that the items of the same register start at the same height of the corresponding slips. Xiao Can notes, however, that in slip /0774/ no suitable space could be inserted between (3.1) and (3.2) to fit this purpose. She thus suggests that the space was replaced by a specific punctuation mark: a bold dot, as opposed to the ‘hook’ which occurs at the beginning of slips /0410/ and /0778/. However, this interpretation of the structure of the text of the table runs into problems.

First, if the layout of the table complied with the principle described above, why do not we have three full upper registers, and only the lowermost register with a single clause, instead of having, as is the case here, two full upper registers and two incomplete lower registers?

Second, a reading of the set of clauses following the principle that we have established above makes it hard to understand the principles according to which the clauses were selected and also arranged.

To understand this point, let us examine the sentences recorded in the cells that are created on each slip by the spaces and the middle chord. Each of these sentences states a multiplication between two operands, and its result. Except for two cases ((1.4) and (2.3)), which are similar to each other, both operands are unitary fractions (that is, of the form \( 1/n \)). Moreover, in all remaining cases but one (1.2), between the statement of the multiplication and that of its result, the clause corresponding to the multiplication of the denominators is quoted from a ‘multiplication table for digits.’ The structure of the sentence placed in a cell in most cases is thus as follows: “\( 1/n \) multiplied by \( 1/p \), \( n \) times \( p \) is \( q \), \( 1/q \)” Let us represent the clauses recorded in the three slips in a tabular layout as follows:
If we read the table following the upper register, we have (1.1) “1/2 times 1/3...,” (2.1) “1/3 times 1/4...,” (3.1) “1/5 times 1/5...” The sequence has no manifest organization, and the same holds true if we continue with the register under this one ((1.2), “1/2 times 1/2”...). If, by contrast, we read the cells slip by slip, we have the same problem. Not only is the organization (and thus the reading to be applied) unclear, but also there seem to be gaps. Indeed, the table seems to record all multiplications between unitary fractions with small denominators, but 1/2 times 1/4, or 1/3 times 1/5, for instance, are missing.

There appears to be one suggestion that might provide a solution to these difficulties. It derives from the observation that the set of sentences (1.1) and (1.2) appears to present a structure similar to the set (2.1) and (2.2). The sequence can be represented as follows ‘1/n times 1/n+1, n times n+1, result of the latter multiplication, result of the former multiplication ye. 1/n times 1/n, n times n, result of the latter multiplication, result of the former multiplication ye.’ Moreover, the other sentences in the table (setting aside (1.4) and (2.3)) all come from the same sequences for n = 4 and n = 5.

This suggests the following hypothesis. Perhaps, in the case of this table, a ‘clause’ is a set of two sentences of the type suggested above, that is, a set of two cells. If this were the case, the text of the table would have two registers (upper and lower), each being divided into two sub-registers. This would solve the problem about the layout that we raised above. In fact, for other tables of Mathematics, clauses also seem to be a pair of sub-clauses. In the case analyzed, I represent this hypothesis by keeping only two registers in the hypothetical tables restoring the original text below.

Now, Daniel Morgan and I have established that in some manuscripts, dots indicated a scribal mistake. The description of the use of dots that we have offered might help us solve the problems raised above with respect to this table. In fact, on the basis of our joint research, here two hypotheses can be formed.

The first hypothesis holds that the dot is placed before a sentence at the end of a section, and indicates that the sentence was omitted above in the process of inscription of the section, and should be inserted earlier in the same section. If this hypothesis holds true for the sentence (3.2), first this

<table>
<thead>
<tr>
<th>Slip 3</th>
<th>Slip 2</th>
<th>Slip 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5 times 1/5...</td>
<td>1/3 times 1/4...</td>
<td>1/2 times 1/3...</td>
</tr>
<tr>
<td>• 1/4 times 1/5...</td>
<td>1/3 times 1/3...</td>
<td>1/2 times 1/2...</td>
</tr>
<tr>
<td></td>
<td>1/3 times 10...</td>
<td>1/4 times 1/4</td>
</tr>
<tr>
<td></td>
<td>1/5 times 1/6...</td>
<td>1/3 times 1...</td>
</tr>
</tbody>
</table>

\[ \text{27 The middle part of the clause is missing for (1.2).} \]

\[ \text{28 See Morgan and Chemla (Submitted).} \]
indicates that this position is perceived to be the end of the section. This implies that, despite a similar layout, the text of Table 2 differs from that of Table 1. Secondly, this hypothesis further raises the question of where the mistake occurred. One might surmise that sentence (3.2) had been omitted at the beginning of the lower register, precisely before the sentence now placed in (1.3), with which it constitutes a set. If, moreover, we assume that after the sentence was omitted, the scribe continued writing the remaining part of the lower register, this would suggest that a global shift was applied to these sentences. Once we restore this sentence to this assumed original position, and undo the related shift in the lower register, some order appears in the table. This thus suggests the following tentative restoration of the structure of the original table:

<table>
<thead>
<tr>
<th>Slip 3</th>
<th>Slip 2</th>
<th>Slip 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5 times 1/6...</td>
<td>1/3 times 1/4...</td>
<td>1/2 times 1/3...</td>
</tr>
<tr>
<td>1/5 times 1/5...</td>
<td>1/3 times 1/3...</td>
<td>1/2 times 1/2...</td>
</tr>
<tr>
<td></td>
<td>1/3 times 1...</td>
<td>1/4 times 1/5</td>
</tr>
<tr>
<td></td>
<td>1/3 times 10...</td>
<td>1/4 times 1/4</td>
</tr>
</tbody>
</table>

The problem with this solution is to account for why slip 3 is perceived as the end of the section if the slip is written register by register. The second hypothesis would be that the clause (3.1) was written in place of “1/4 times 1/5...” The mistake was signaled by a dot, and corrected immediately after. The scribe then went on with writing, in the lower register. What happened next is open to conjecture.

Either the scribe went on writing down, and in the end did not repeat the sentence mistakenly written. This would suggest the following structure for the original text:

<table>
<thead>
<tr>
<th>Slip 3</th>
<th>Slip 2</th>
<th>Slip 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4 times 1/5</td>
<td>1/3 times 1/4...</td>
<td>1/2 times 1/3...</td>
</tr>
<tr>
<td>1/4 times 1/4</td>
<td>1/3 times 1/3...</td>
<td>1/2 times 1/2...</td>
</tr>
<tr>
<td></td>
<td>1/5 times 1/6...</td>
<td>1/3 times 1...</td>
</tr>
<tr>
<td></td>
<td>1/5 times 1/5...</td>
<td>1/3 times 10...</td>
</tr>
</tbody>
</table>

Or the scribe skipped the set of sentences related to 1/5, and placed the only missing sentence of its set at the end of the text. This possibility leads one to restore the original text as follows:
In the context of the latter two hypothetical restorations of the text of the table, the end of the table would actually be in (2.4), and the reading (and writing) of the table would be identical to the one established for Table 1 (except that now clauses are composed of two subclauses). We will see, however, that the same type of clues as those used above show that for tables written down in the same way in manuscripts, the material format was not always read as was established above for Table 1.

Whatever the original text might have been, these remarks suggest that in this case, we must probably rethink the nature of the clauses composing the table: the clauses appear to have been composed of two cells. The fact that the space between registers 1 and 2 was not always clearly marked might be a consequence of this fact.

A Second Type of Text for the Same ‘Table-Relation’

Table 1 examined above was a ‘multiplication table for powers of ten’, and we have seen that on the slips its clauses were recorded in the cells formed by registers. In other words, they all ended with the particle ye 也 and a space. Interestingly enough, Writings on Mathematical Procedures includes a table of the same type, that is, a table corresponding to the quantitative relation of multiplication, for powers of 10. However, among the various differences that the two tables present vis-à-vis one another and that we will examine in turn, one relates to the way in which they are textualized. As I will show, the text of the table in Writings on Mathematical Procedures is materialized on the slips in a completely different way. To introduce the phenomenon, I first quote and translate the text, giving a word-by-word translation of the numerical values between parentheses for the sake of the analysis to be developed. The table (Table 3) reads as follows:

<table>
<thead>
<tr>
<th>Slip 3</th>
<th>Slip 2</th>
<th>Slip 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4 times 1/5</td>
<td>1/3 times 1/4...</td>
<td>1/2 times 1/3...</td>
</tr>
<tr>
<td>1/4 times 1/4</td>
<td>1/3 times 1/3...</td>
<td>1/2 times 1/2...</td>
</tr>
<tr>
<td></td>
<td>1/3 times 1...</td>
<td>1/5 times 1/6...</td>
</tr>
<tr>
<td></td>
<td>1/3 times 10...</td>
<td>1/5 times 1/5...</td>
</tr>
</tbody>
</table>

Note that I have indicated the punctuation marks that the written text contains. The hook mark (∟) is often used to separate between sentences, while the ligature (＝) indicates the repetition of the character after which it is placed. For the edition of the text, see Peng Hao 彩浩 (2001), pp. 41-42, Chôka zan kankan Sansûsho kenkyûkai (2006), pp. 149-150, and plate 22. Note that the table begins after a space on what is believed to be the previous slip. However, it does not begin with a title. Guo Shuchun 郭書春 (2001), p. 204, note 1, considers that this table must have constituted the first paragraph of the section, not the second, as the slips now have it. Moreover, Guo Shirong (2001), p. 278, and Guo Shuchun (2001), p. 204, note 1, both consider that the
一乘十$a_{n}$，(十)也$L$；十乘万，十萬也$L$；千乘萬，千萬。  
一乘十$5a_{n}$，(十萬)也$L$；十乘十萬，百萬$L$。半乘千，五百$L$。一乘百$a_{n}/$slip 11/ 
万$a_{n}$，(百萬)也$L$；十乘百万，千萬$L$。半乘万，五千万$L$；十乘千，萬也$L$；百乘万，百萬$L$；半乘百，五十。  
(space) /slip 12/ 

One multiplying ten is ten$L$; ten multiplying ten-thousand is a hundred-thousand (ten ten-thousand)$L$; a thousand multiplying ten-thousand is ten millions (a thousand ten-thousand). One multiplying one hundred thousand (ten ten-thousand) is one hundred thousand (ten ten-thousand)$L$; ten multiplying one hundred thousand (ten ten-thousand) is a million (a hundred ten-thousand)$L$; half multiplying a thousand is five hundred$L$; one multiplying a million (a hundred /11/ 

ten-thousand) is a million (a hundred ten-thousand$L$; ten multiplying a million (a hundred ten-thousand) is ten millions (a thousand ten-thousand)$L$; a half multiplying ten-thousand is five thousands$L$; ten multiplying a thousand is ten thousand$L$; a hundred multiplying ten-thousand is a million (a hundred ten-thousand)$L$; a half multiplying a hundred is fifty. /12/  

The text of Table 3 consists clearly of a sequence of parallel clauses that are all stating results of multiplication between powers of 10, except for three clauses for which one operand is ½. It is a frequent phenomenon that in tables about integers, one also finds the value ½. With the definition we gave for the expression, we clearly have a ‘table-relation,’ the text of which is not written with a ‘table-layout,’ but as ‘a textual table.’ As in the other ‘multiplication table for powers of ten,’ for each clause the first operand is smaller text originally recorded first all clauses whose first term was 1, then all clauses whose first term was, respectively, 10, 100, and ½. None of them features the punctuation marks in the text of the table. However, Guo Shirong 郭世荣 (2001) considers that all clauses ended with the particle ye 也, and suggests restoring it everywhere it was missing.  

30 This is the only clause among those recorded on the two slips that does not end with the particle ye 也 and a punctuation mark.  

31 From here on, all the clauses of the table end with a punctuation mark, but, except for one, they do not end with the particle ye 也.  

32 Here the punctuation mark $L$ was omitted.  

than the second, and no result is larger than ‘ten millions,’ that is, “one thousand ten-thousand qian wan 千萬,” for which one would have needed the introduction of another term.

The table-relation is, however, textualized using not registers, as above, but continuous text with clauses generally separated by punctuation marks, and occasionally by the particle ye 也 before a punctuation mark. In fact, all table-relations in Writings on Mathematical Procedures, without exception, and only tables, are textualized in the same way (that is, as lists of clauses with the repetition of punctuation and the particle ye 也 — when clauses have sub-clauses, punctuation marks also separate sub-clauses). Moreover, the only table of Mathematics that is not written with registers is textualized with recurring punctuation marks.\textsuperscript{34} This table is also found in Writings on Mathematical Procedures, where it is textualized in the same way.\textsuperscript{35}

These remarks lead to several consequences.

First, this enables us to establish that, even though the documents discussed so far contain no actor’s term referring to this specific type of text, there must also have been an actor’s category corresponding to the observer’s category ‘table-relation’. We return to this issue in the following part of this article.

Second, we can rely on these conclusions to establish the items of knowledge expressed in table form in a manuscript. For instance, in Writings on

\textsuperscript{34} This is in the context of the table-relation for the topic Shaoguang 少廣 ‘Reducing width.’ See below, and also Mo and Lin (2016). The text of the slips is reproduced and annotated in Zhu Hanmin 朱漢民 and Chen Songchang 陳松長 (2011), pp. 23-24, 120-124. See also Xiao Can 蕭燦 (2010), pp. 81-83; id. (2015), pp. 98-102. On pp. 10-11, Xiao Can (2010) (that is, Xiao Can (2015), p. 14) describes the punctuation signs used in Mathematics. In her view, the punctuation sign we mention is only used to disambiguate, and she does not comment on the systematic recurrence of the sign in relation to writing textual tables. Accordingly, she describes the use of the punctuation signs as similar in Mathematics and Writings on Mathematical Procedures. In my view, this is only partly correct. First, even though she is absolutely right to mention a use of the hook sign to disambiguate, it is not its only use. Moreover, many tables in Mathematics are not textualized using punctuation marks, and hence the use of the hook mark is not wholly identical in the two manuscripts.

\textsuperscript{35} Liu Jinhua 劉金華 (2003), p. 27, noted that among all multiplication tables found in Han manuscripts from Dunhuang and Juyan, only one was not written with registers, but continuously. It is slip 36.5 from Juyan (see Xie Guihua, Li Junming, Zhu Guozhao (1987), p. 58). According to the critical edition, the slip does not have any punctuation mark separating clauses, but it begins with a dot. This corpus enabled Liu Jinhua to identify one way of writing tables in manuscripts, but not the second one that we have just described.
Mathematical Procedures, in addition to Table 3, translated above, tables include: tables related to the computation of volumes,\textsuperscript{36} areas\textsuperscript{37} and the multiplication between fractions (slips 3, 5, 6, and then slips 8 to 10); tables for converting between (non-decimal) measurement units of weight (slip 47); tables recording equivalences between fractions of different grains, with procedures computing them (slips 98-104); tables recording the equivalences between unhusked and husked grains, depending on the loss in capacity due to the husking operation (slips 105-108); tables of procedures to compute equivalences between different types of grains (slips 109-110, slips 111-112); tables for reducing fractions to a same denominator (slips 167-181).

Even though these tables and the table-relations included in Mathematics presented differences (in particular since Mathematics contains more tables recording data about grains), the topics of the tables are to a great extent the same in the two manuscripts.\textsuperscript{39} Two common features must be stressed in this respect. In addition to tables related to multiplicative operations and conversion (which could also be considered to fall under the former label), the topic of grains, and only this topic, is massively represented in the table-relations. Moreover, in both manuscripts, we find procedures among the terms occurring in the clauses of tables. In such cases, the clauses also regularly have multiple sub-clauses.

\textsuperscript{36} See slips 1, 2, 4, 5 (I do not discuss editorial problems here). On the basis of an observation of the actual use of measurement units in problems and procedures, Chemla and Ma (Forthcoming) establish that the first table deals with the computation of volumes and not areas, as has been so far asserted. This illustrates a method I suggest using to determine what practitioners actually did with these ‘table-relations.’

\textsuperscript{37} See slip 8. In fact, the actual presence of this type of clause is not certain, depending on the editorial decision one takes about the presence of the measurement unit by \textit{bù}. I do not dwell on this issue here and will return to it in another publication.

\textsuperscript{38} I am grateful to Zhu Yiwen, whose remarks after my talk helped me grasp this also as a table. Mo (Morgan) and Lin (Chemla) (2016) rely on this identification to offer a hypothesis on the nature and meaning of the document Writings on Mathematical Procedures.

\textsuperscript{39} Differences in the sets of table-relations that the two manuscripts contain and in the ways of shaping clauses and sets of clauses for the same quantitative relation await further research. I plan to return to it in a future publication. Notably, the table-relations for converting among weight measurement units are all different, and it would be important to clarify what this feature tells us about the nature and the use of such a table and on the manuscripts containing them. Guo Shirong 郭世荣 (1998) analyses the computations that were required for stating numerical values that are found in excavated documents from early imperial China. Several of these computations echo the tables found in mathematical manuscripts. The relationship between the two corpuses of documents also awaits further research.
A second material feature distinguishes the slips recording tables from the others. As Peng Hao 彭浩 (2001), p. 3, notes, three signatures appear at the end of some slips (usually the first slip of a section or of a paragraph in a section). Peng suggests that these are the names of persons checking the correctness of the text written in a section. The signature Wang 王 occurs three times, Jing 聶 once, and the signature Yang 楊 eleven times. Only Yang signs tables. Moreover, out of the 11 paragraphs (for 7 sections) recording tables, 8 are signed by Yang (Table 3 is thus one of the rare paragraphs receiving no signature; two other signatures by Yang are on damaged slips, and the third is an unusual one). These remarks are certainly essential to gaining a better understanding of the process in the context of which the mathematical manuscript was produced. I plan to return to this issue in another publication.\(^\text{40}\)

Third, the conclusion drawn about the textualization of table-relations suggests that, seen from this perspective, Mathematics is not homogeneous. This might either be an indication of a feature of the manuscript worth exploring further, or a hint that in some contexts, different types of text for tables reflect different functions.

Finally, these remarks suggest that the two ways in which tables were textualized might have enjoyed some stability, and at the very least that they were not an idiosyncrasy. If this holds true, this means that Table 3, the text of which is translated above, appears to have no mistakes in its clauses, except that one punctuation mark was dropped. As a result, the text of the table-relation was not correctly realized.\(^\text{41}\)

\(^{40}\) From a completely different perspective, Mo and Lin (2016) exhibits another material feature in the section Shaoguang 少廣 ‘Reducing width’ of Writings on Mathematical Procedures that further confirms that the opposition between texts of tables and continuous text was an actor’s distinction. Interestingly, this is precisely the topic for which Mathematics uses the type of text with hook marks for a table-relation.

\(^{41}\) These conclusions are supported by the following remark. Some dots that feature in slips in which tables are recorded in Writings on Mathematical Procedures or in Mathematics apparently signal mistakes in the way of realizing the text of the table. See, for instance, slip 1135 (editors’ number 89) of Mathematics, in Zhu Hanmin 朱漢民 and Chen Songchang 陳松長 (2011), p. 14, or slip 98 of Writings on Mathematical Procedures, in Chôka zan kankan Sansûsho kenkyûkai (2006), plate 22. Daniel Morgan’s paleographic analysis of the latter document has established that the two paragraphs composing the section to which slip 98 belongs were written by two hands, the first by hand A (less experienced) and the second by hand B (more experienced) (see Mo and Lin (2016)). In fact, it appears that the second paragraph contains a textual table the text of which is realized in a much stricter way than the first paragraph, which contains the dot. A difference between these two paragraphs and Table 3 (which also apparently contains a mistake in the realization of the table) is that the two paragraphs both contain the signature of Yang 楊, whereas Table 3
In comparison to the ‘multiplication table for powers of ten’ translated above (Table 1), the order of the clauses of the table contained in *Writings on Mathematical Procedures* does not show clear order, and most of those who worked on a critical edition of the manuscript thought it was garbled and suggested that it be reordered (see footnote 29). However, one might make the following remark. Suppose that 10000 (wan 萬) has been perceived as a large unit. The structure of the table can thus be represented as follows (noting only the operation, and not the result, and also inserting line feeds):

1 times 10

10 times 1 unit wan

*(missing)* 100 times 1 unit wan *(occurs at the end)*

1000 times 1 unit wan

*(missing perhaps here)* 1/2 times 100 *(occurs at the end)*

1 times 10 units wan

10 times 10 units wan

1/2 times 1 000

1 times 100 units wan

10 times 100 units wan

1/2 times 10 000 *(one unit wan)*

10 times 1000

100 times 1 unit wan *(clause perhaps missing above)*

1/2 times 100 *(clause also seems to be missing above)*

If the last two clauses had been omitted earlier (note that often, omitted bits are added at the end of paragraphs), and one formed the hypothesis that 1 was considered a small unit while 10 000 was a large unit, then this reading might manifest a greater order than was originally perceived. This explains why I mentioned above the literal translation of numerical values. Likewise, the perception of the table as lacunary or not (by comparison with the first table) also depends on the reading we make. Whatever the case might have been, it seems plausible that the function of the two tables was not the same.

Let us for now summarize our conclusions on the types of text with which table-relations were textualized in the manuscripts that we have discussed above. We have seen that the mathematical manuscripts examined so far...
contained only textual tables, and no table-layout to write down table-relations. Moreover, we have established that two specific types of textual form were used to write down table-relations: one made use of registers, while the other made a specific use of punctuation marks. Both types of textual form seem to have been written continuously, even though the movement of the writing instrument on the writing support might not have always been from top to bottom and then from right to left. Were these table-relations copied? Were they produced out of texts learnt by heart? Were they the result of other forms of activity, like exercises? Is the organization of clauses that some of these tables display related to a use like pick-up data? For the moment, it is not possible to answer these questions.

Finally, it may be that tables in Qin-Han manuscripts tend to have been written down in a uniform way. This hypothesis must be tested against a greater number of cases. Examining it further will help us understand better whether the type of text used to write down a table is a matter of specific manuscript cultures, or whether it conveys a meaning that has so far escaped me. In any event, Mathematics already represents an exception to this rule, since one of its tables was written in a way different from all the others. Understanding why this is so will help us grasp the nature of this document and to go deeper into understanding actors’ categories for tables.

Calculating Tools with Table-Layout

As can be seen from Liu Jinhua 刘金華 (2003), manuscripts from the Qin and the Han time periods abound in ‘multiplication tables for digits.’ Except for one Juyan slip, which shows succeeding clauses of a table written from top to bottom on a same slip, what remains of these tables seem to indicate that most of them were written continuously, register after register, from the top down. Moreover, several among these documents, like the Liye example discussed above, end the text of a ‘multiplication table for digits’ by stating the sum of the results of all clauses. As far as I can tell on the basis of the published documents, two sums are attested: 1110 and 1113, depending on the end part of the table. This feature further reinforces the conclusion that, in the eyes of the scribes, these tables constitute a single piece of text.

A document written on a wooden board found in a site from the eastern Han dynasty at Zhangjiajie (Hunan) seems to attest, however, to the use of another way of inscribing the same table-relation. Indeed, the legible part of the text suggests that, although the inscription of the table-relation still made use of registers, these registers were filled up in a different way, since eventually, the table (to which I will refer as Table 4) seems to have been

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42 Hunan sheng wenwu kaogu yanjiusuo 湖南省文物考古研究所 et al. (2003a), p. 76. What follows relies on the transcription given in the article, since I could neither consult the document, nor find a photo of it.
written using a global triangular layout as follows (I only write the operands of the clauses, not their results, and I represent the space of a clause by that of a cell; the grey color indicates clauses, or parts of a clause, that are not legible: I indicate some of the plausible restorations to highlight the structure of the whole; finally, I mark two clauses in bold characters, since I will refer to them below):

<table>
<thead>
<tr>
<th>2 times 9</th>
<th>3 times 9</th>
<th>4 times 9</th>
<th>5 times 9</th>
<th>6 times 9</th>
<th>7 times 9</th>
<th>...</th>
<th>9 times 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 times 8</td>
<td>3 times 8</td>
<td>4 times 8</td>
<td>5 times 8</td>
<td>6 times 8</td>
<td>7 times 8</td>
<td>...</td>
<td>8 times 8</td>
</tr>
<tr>
<td>2 times 7</td>
<td>3 times 7</td>
<td>4 times 7</td>
<td>5 times 7</td>
<td>6 times 7</td>
<td>7 times 7</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>2 times 6</td>
<td>3 times 6</td>
<td>4 times 6</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>2 times 5</td>
<td>3 times 6</td>
<td>4 times 5</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
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<td></td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2 times 2</td>
</tr>
</tbody>
</table>

If consultation of the document itself confirms the layout, this document might constitute the earliest extant example of the use of a table-layout (in observer’s terms) to write down a numerical table-relation in Chinese documents. In this case, several questions present themselves. Why did the scribe adopt this layout? Was this related to a reflection, or an operation, on the table that would be facilitated in this way? Or, did this layout derive from how the table was produced? Or else, was this the result of the expectation that the layout could help pick-up data? These questions need to remain open, until we get more information on the document and perhaps also

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43 We cannot yet observe photos of the table included in Part A of *Suanshu* 算書 (Mathematical Writings), contained in the fourth set of Qin slips on mathematics kept at Peking University. However, the fact that it is written on 8 slips with 5 registers seems to rule out the possibility that it has a similar layout (Han Wei 韓巍 (2012), p. 3)
determine whether clues in it can help us to answer them. However, the description of another piece of evidence might help us to gain further insight.

In fact, my tentative conclusion that the Zhangjiajie document might be the earliest example of a table-layout to write down a table-relation among Chinese documents seems to be contradicted by a further recent discovery. The layout described above immediately evokes a Qin artifact that was the product of illegal excavation and is now kept at Qinghua University (Beijing). Its editors, who give it the title “Calculating table” (suanbiao 算表), believe that it dates from even before the establishment of the Chinese empire. This artifact is composed of a set of 21 bamboo slips that are longer and wider than those commonly used for writing. The slips form a double-entry table-layout, the cells of which are marked by red horizontal lines drawn across the set (see a partial reproduction on Figure 3). The lines divide slips into cells that record results of multiplication between operands, one of which is on top of the slip in which the cell occurs, while the other operand lies to its right, on the slip placed to the rightmost position. The operands run from tens (90, 80, ... 10), to units (9, ..., 1), to end, again, with the fraction \( \frac{1}{2} \). The presence of the fraction confirms what we have seen above: \( \frac{1}{2} \) regularly occurs in the context of many table-relations dealing with integers.

This table-layout writes down a table-relation. The text of a clause in this way of writing down the table-relation is distributed on the space formed by the set of slips: one operand occurs on the rightmost slip, the other operand in a top position, and the result in a central cell. By contrast to all the multiplication tables we have described above, the artifact does not limit the number of clauses by half, using the symmetry of the multiplication (both the clause \( n \times m \), and the clause \( m \times n \) occur). This feature might be correlated with the fact that this artifact was in fact used as a computing tool, meant to yield results.

This conclusion derives from other material features of the set of slips. On the right hand side and on top, the set shaped by the slips also has holes, in which evidence of silk threads remains, indicating that threads were attached to the holes and helped the user to get the results. The editors argue that the object might have been used to perform multiplications, and perhaps also divisions, between numbers smaller than 100 and also possibly having the fraction \( \frac{1}{2} \). The operations would have been broken up into elementary operations, the results of which could be picked up from the table and added together to yield the final result. According to

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44 See photos and analyses in Qinghua daxue chutu wenxian yanjiu yu baohu zhongxin 清華大學出土文獻研究與保護中心 and Li Xueqin 李學勤 (2013); Li Junming 李均明 and Feng Lisheng 冯立昇 (2013), pp. 73-75.

45 Li Liang (personal communication) notes that the use of symmetry properties in tables employed in the context of the astral sciences is rather late. This difference might point out a difference of nature between the tables that awaits further analysis.
this highly plausible hypothesis, the object would thus have required specific handling. Moreover, the choice of the clauses to include on the object reflected this handling.

These remarks have important consequences. The fact that this table-relation is written on the same physical materials as that used for writing should not deceive us: in fact, I argue that this Qin artifact is an object, that is, a material calculating tool, rather than a written document, comparable to those examined above (Chemla Forthcoming). The table-layout is directly linked with this function, since silk threads rely on vertical lines and horizontal lines to determine the numbers to be picked up. This retrospectively explains why I have not included this artifact in the corpus of table-relations occurring in Chinese writings, the textual forms of which I discussed above. In this case, the use of the tabular layout seems to have been correlated to the nature of the artifact as an object and its function as a tool.

This Qin object echoes a Tang dynasty Chinese document from Dunhuang (Pelliot chinois 2490, see Figure 4), which records a table that was written down in 952 and shares several features with this “Calculating table.” Made with paper of rather large size, it is composed of pages attached to a wooden pole. Its sheets are also divided into cells marked by red lines, and in the cells, area measurement units are used to record the results of multiplication between lengths, placed virtually on top and on the right hand side (except that some top entries are marked on protruding paper strips). The structural and material similarity between the Qin table and the Tang table is thus striking. The Dunhuang table might actually also be interpreted as a calculating tool, produced with paper, rather than a text stricto sensu. This would point out another similarity, between the two objects, since in both cases, a material support for writing (in one case, wooden slips, in the other, paper) was used to make tools for practice.

In both cases, thus, the use of a table-layout can be related to the function of the inscription of the table-relation as a pick-up device. This discussion enables us to return to our Zhangjiajie table-relation and hone our questions about it. Should its tabular layout be confirmed, might its vertical and horizontal lines have served similar functions? Might the object itself likewise have served as a pick-up device, even though its clauses were written differently from those in the objects evoked above? In the case that it was not itself a tool, would the document attest to the transfer of pick-up tables from objects to texts? In the context of this discussion, it is interesting that this document does not seem to contain the sum of the results of all clauses, which suggests that its clauses might not have been perceived of as constituting a whole.

Answers to these questions will help us determine when exactly, among the mathematical writings so far known, tabular layouts began to be used to write down a table-relation.
Among Loulan documents that are dated from the end of the third century and beginning of the fourth century, we find two fragments of a ‘multiplication table for digits’ that represent the earliest known table-relations written on paper in Chinese.\(^{46}\) Hu Pingsheng 胡平生 has suggested these fragments derived from a text of a table comparable to that from Zhangjiajie mentioned above (Table 4).\(^{47}\) However tiny they may be, what the fragments show indicates that this hypothesis is highly plausible. In such a case, again, these fragments might again constitute a first known example of a table-layout used to write down a table-relation on paper. However, two clues seem to indicate that more reflection is still needed before one might draw this conclusion with certainty. Fragment 22,15, indeed, is identical to the upper left corner of Table 4, schematized above (see the two clauses marked with bold characters in the schematized representation of the table). However, ink seems to appear below the leftmost clause, suggesting the end part of the table (or of all sets of clauses in a column) might present differences. More difficult is the fact that there is a space below the single clause ‘nine ninety-one’ that appears on Fragment 22,16. This feature is not easy to account for in the context of Hu Pingsheng’s hypothesis. Probably new documents will help us better interpret these fragments, and I leave the issue here for the moment.

We have now an overview of the types of text used to write down table-relations in the earliest known mathematical documents. On this basis, we can examine continuities and discontinuities in how table-relations were textualized in later manuscripts (from central Asia) and in early printed editions, as well as in later works.

**Exploring Continuities and Discontinuities in Later Documents**

**Texts for ‘Table-Relations’ in Later Manuscripts**

Table-relations occur significantly among other Dunhuang manuscripts that are devoted to mathematics and attest to a widespread use of paper for mathematical practice.\(^{48}\) In fact, the texts of these tables are in close continuity with what we have described for the earlier manuscripts, as if, in this

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\(^{46}\) See documents 22,15 and 22,16, in Conradi (1920), p. 102 (for the transcription) and plate XXV (for a reproduction).

\(^{47}\) Hunan sheng wenwu kaogu yanjiusuo 湖南省文物考古研究所 et al. (2003b), pp. 70-71.

\(^{48}\) These mathematical documents are edited in Li Yan 李巖 (1954). For an analysis, see Libbrecht (1982). For all Dunhuang manuscripts, I also rely on the online
case too, the change of material support did not affect writing practices for inscribing table-relations.

Manuscript *Pelliot chinois 3349*, which bears the title *Suan jing* (Mathematical Canon), and seems to date from the second half of the tenth century, is an interesting case. To begin with, its recto contains a ‘multiplication table for digits’ (see Figure 5). I will not comment on all the tiny differences between the text for this table and the texts featuring the same table-relation encountered above, limiting myself to differences that are essential for my purpose. As was the case for the Qin board found at Liye, the table-relation is written in registers, in this case four registers. Seven columns of writing and the three horizontal spaces running across them define cells in each of which a clause is inscribed. However, the arrangement of the clauses makes it clear that although the layout of the text is the same, it does not correspond to the same use: now, clauses are read/written first from top to bottom, and only then from right to left. Accordingly, the writing of the table is more regular than was the case for the Liye board (in which the number of clauses varied from one register to the next).

In addition, in *Pelliot chinois 3349*, in some of the spaces after a clause, a hand has inserted the representation of the numerical value of the result of the related clause using counting rods. This is one of the two earliest illustrations of the use of rods to represent numbers in a Chinese document. The other is also found in a Dunhuang manuscript (*Or. 8210/S.930*), where the system is first explained using an illustration, and then likewise used in the context of the statement of a textual table that contains a ‘multiplication table for digits’ (see Figure 6). The number system fits with the verbal description of how to use rods to represent numbers, given prior to the table in the manuscript titled *Suan jing* (Mathematical Canon by Master Sun), the completion of which Qian

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notice published with the documents on the International Dunhuang project website (the URL are given in the references).

49 On this manuscript, and the editions of the mathematical writing it contains, see Li Yan (1954, p. 33), Libbrecht (1982, pp. 204-205). Whereas the manuscript seems to bear traces dating to the second half of the tenth century, Li Yan (1954, pp. 52-54) dates the mathematical part of text that the manuscript contains to the Tang dynasty. This latter date might be subject to revision. Guo Zhengzhong 郭正忠 (2001) has shown that Li Yan’s date for *Pelliot chinois 2667* raised problems, and he has suggested a much earlier date for the writing.

50 On this manuscript, and the editions of its mathematical part, see Li Yan (1954, p. 33), Libbrecht (1982, p. 205). Again, Li Yan (1954, pp. 54-55) dates the mathematical text that the manuscript contains from the Tang dynasty, but this might require re-examination, along the lines suggested in Guo Zhengzhong 郭正忠 (2001). As for the manuscript, it bears traces as late as the second half of the ninth century.
Baocong dates to ca 400. 51 This suggests that the description of the number system in the Sunzi suanjing can be interpreted as referring to what the Dunhuang manuscripts show. As we will discuss below, the perspective adopted in this article shows that the first chapter of the Sunzi suanjing is mainly constituted of table-relations. In these later documents, we thus identify a correlation between discussing rods, illustrating rods, and writing table-relations. In any event, in the context of the two Dunhuang manuscripts, the texts of the clauses now include an additional element that relates to the material tool of computation that rods constituted. The fact that representations of numbers with rods first occur in writings of this type, and in relation to the text of tables and only there, also distinguishes tables from the discursive parts of texts. This phenomenon awaits further research.

In Pelliot chinois 3349, the ‘multiplication table for digits’ is followed by another table, likewise written as a textual table of a register type (with two registers), and embedding the same ‘multiplication table.’ In this second table, however, each column of text contains a single clause, and each clause has two sub-clauses (this is yet another illustration of the phenomenon described in footnote 3). The upper sub-clause repeats a clause from the multiplication table, whereas the lower sub-clause states an operation based on the result of the upper one and yields its result. Also after the \( n \)-1 clauses corresponding to the multiplications of 2, \( \ldots \), \( n \) by \( n \), the sum of the results of the upper sub-clauses are added. Since in Pelliot chinois 3349, the ‘multiplication table for digits’ was given before in the text, the point of the new table is certainly not to state it, but rather to rely on the table to carry out mathematical operations. In fact, the manuscript Or. 8210/5.930 gives the ‘multiplication table for digits’ in the context of a wider table with a comparable structure. The Dunhuang manuscript Or. 8210/5.19 also contains a table-relation of the same type (see Figure 7), in which, however, clauses have (at least) three sub-clauses (the lower and left parts of the document are damaged), and the two lower sub-clauses record operations (and results) based on the results of the upper sub-clauses. The operations carried out in the lower sub-clause(s) differ in the three manuscripts, but the use of the upper table is the same. 52 This suggests that table-relations were used as a support for carrying out operations, giving us a first insight into the use of table-relations in these contexts. After all, a similar use is evidenced through the statement of the sum of all results of clauses that we find in many Qin-
Han tables of the same type. Such uses of tables are quite specific. Mo and Lin (2016) establish that other table-relations were used in the same way in Han mathematical manuscripts. As a result, we do perceive continuity between Qin-Han documents and Dunhuang manuscripts not only in terms of textual tables of register type, but also in uses of table-relations.

In fact, continuity seems to extend to the other type of textual tables identified in Qin-Han manuscripts. Indeed, the three Dunhuang manuscripts, Pelliot chinois 3349, Or.8210/S.19 and Or. 8210/S.930, discussed above, further contain table-relations stating relationships between measurement units or relationships between names for large numbers. Interestingly enough, the textual form used for these table-relations is the same in the three documents: clauses, or sets of clauses, separated by the empty space of a character. However, since the clauses do not have the same length, these spaces do not constitute registers. This remark leads to several conclusions. First, in these later manuscripts, if we set aside manuscript Pelliot chinois 2490, all table-relations are thus written using textual tables, and not diagrammatic tables. Secondly, it seems that in this manuscript culture, the empty space of a character left between clauses was playing a part similar to that played by the hook mark in the Qin-Han manuscripts evoked above. Finally, as was the case for Mathematics, we find in Dunhuang manuscripts two different forms of textual tables used conjointly for different table-relations. What this tells us about the meaning of the layout is an open question.

**Texts for ‘Table-Relations’ in Early Printed Books and Song-Yuan Works**

The earliest known printed editions of mathematical writings likewise testify to an important continuity with the later manuscripts examined above, with respect to both the textualization and the uses of the tables. These printed editions are the early thirteenth-century publications of the anthology Ten Canons of Mathematics by Bao Huanzhi 鮮澤之. Upon imperial order, together with some colleagues, Li Chunfeng had selected these canons and ancient commentaries, edited them and commented upon them, presenting the resulting anthology to the throne in 656. These canons were shortly afterwards used as textbooks in two mathematical curricula taught in the “School of Mathematics suanxue 算學.” Bao Huanzhi reprinted the

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53 In Chemla and Li (Forthcoming), Li Liang provides evidence that the same way of writing down tables is found among Dunhuang astral documents (Pelliot chinois 2512).

54 The document has been reprinted in Shanghai tushuguan and Beijing daxue tushuguan (1980).

55 On the curricula in the school, see Volkov (2014).
first 1084 printing of these canons, adding an auxiliary text used in teaching, which he retrieved from a monastery.

The first remark concerning our topic that the texts of these canons inspire is the relative scarcity of table-relations in them, by comparison with the manuscripts. The two oldest canons from the Han dynasty, *Jiuzhang suanshu* 九章算術 (The Nine Chapters on Mathematical Procedures) and *Zhoubi* 周髀 (The Gnomon of the Zhou) each contain a single table-relation. At the beginning of chapter 2 (Song edition, *juan* 2, pp. 1a-1b, see Figure 8), *Jiuzhang suanshu* presents a table of equivalence among grains, which is written using two registers. Here too, the arrangement of the clauses makes it clear that its reading is identical to the reading described for Dunhuang manuscripts. The *Zhoubi* has a table of lengths of gnomon shadows at various moments of the year (Song edition, *xia juan*, pp. 12a-12b, see Figure 9), the text of which records a single clause per column.56

A key exception to the statement about tables in canons must, however, be mentioned: it is precisely the first chapter of the *Sunzi suanjing*, which consists mainly in tables. The tables it contains include: tables of conversion between measurement units; tables about names of large numbers; table of specific weights of stones and metals; tables of procedures for computing equivalent amounts of different grains; tables of operations involving fractions; a table based on the ‘multiplication table for digits’ seemingly quite close to those contained in Dunhuang manuscripts *Pelliot chinois 3349* and *Or. 8210/S.49*. The topics of these tables are on the whole strikingly similar to what we find in Qin and Han mathematical manuscripts. In particular, the two features I have emphasized above about the tables in the early manuscripts are still present: first, in addition to tables related to multiplicative operations, including conversion between different measurement units, grains are still a notable topic, and, second, clauses regularly have terms consisting of procedures.

In the Song edition of the *Sunzi suanjing*, these table-relations are now textualized either as continuous text (no space or any other mark separating clauses) for the former two tables, or with register-type texts for all the others. However, for the last table, the layout and size of characters are such that a clause now requires two columns (see Figure 10). Whether we consider the textualization, the use, or the contents of tables, we thus see important continuities between manuscripts and early printed texts.

Volkov (2014) has emphasized that the *Sunzi suanjing* was the first canon taught in the most elementary of the two curricula. What we just described could thus mean that table-relations played a major role in the basic part of mathematical education. This fact might shed further light on the nature of

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56 In Chemla and Li (Forthcoming), Li Liang shows a similar pattern for the early printed editions of the monographs devoted to astral sciences in the dynastic histories.
manuscripts in which there are numerous table-relations. What is more, in the *Sunzi suanjing*, like, in fact, in a Dunhuang manuscript like *Pelliot chinois 3349*, which is relatively long and written on a continuous support, tables are all gathered at the beginning of a book. This remark suggests that table-relations thus might have marked the early stages of elementary learning. Interestingly, Part A of *Suanshu* (Mathematical Writings), which is included in the fourth set of Qin slips devoted to mathematics and kept at Peking university, has two tables placed immediately after the dialogue on mathematics that constitutes its first part: a ‘table of multiplication for digits’ (for Han Wei, its section 2) and the procedure *Shaoguang* 少廣 ‘Reducing width,’ which we have encountered above (for Han Wei, the beginning of its section 3; see Han Wei 韓巍 (2012), pp. 2-4, and footnotes 34 and 40 above). The presence of the dialogue and the position of the tables might both indicate the relationship between these slips and mathematical education.\textsuperscript{57} We will nevertheless see that the opening part of a writing is a recurring position for tables.

Another point is worth mentioning about these sources: with a Dunhuang manuscript like *Or.8210/S.930* and the *Sunzi suanjing*, we find the earliest known actors’ categories referring to tables, or to their components. Indeed, we have mentioned that Dunhuang manuscripts and the *Sunzi suanjing* shared large tables based on the ‘multiplication tables for digits,’ and that these tables included an operation of adding all results of the upper sub-clauses after the $n-1$ (or $n$) clauses beginning with the multiplications of (1), 2, ..., $n$ by $n$. Interestingly, the *Sunzi suanjing* refers to these sub-clauses by the term tiao 条 (e.g., Song edition, p. 6a), attesting to the fact that the table-relation is perceived of as being composed of lists of similar types of statements that the term designates.

Moreover, the title of the mathematical part of Dunhuang manuscript *Or. 8210/S. 930* refers precisely to the fact that it contains tables: *Licheng suanjing* 立成算經 (Mathematical Canon with Tables). In the context of the astral sciences, the term *licheng* seems to appear after the Tang dynasty and at the beginning to designate pick-up tables meant to ease computation, before it came to designate ‘mathematical tables in general.’\textsuperscript{58} The title of the Dunhuang mathematical text thus attests to the fact that the term was also used in the context of mathematics. The Dunhuang manuscript *Or.8210/S.930* only has the table-relation with operations that was described above. It is thus this type of table-relation, written down as a textual table, which is referred

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\textsuperscript{57} We must wait until the documents are published to ascertain this conclusion. Indeed, what for Han Wei (2012, p. 4) constitutes the fourth and last section also seems to be a set of clauses that could have constituted a table. It is interesting that apparently, it is partly written in empty spaces left in other writings recorded in the same set of slips.

\textsuperscript{58} See Li Liang (2016, this issue), footnote 25.
to with the term licheng. The table-relation does not seem to have the function of being a pick-up table, hence the occurrence of the term might testify to the fact that licheng had at the time a general meaning of numerical table in the context of mathematical writings. This term licheng constitutes a second actor’s category with respect to tables.

Interestingly, this term (licheng) was employed, probably in the eleventh century, to name one of the two methods used at the time to execute root extraction. As Qian Baocong (1966), p. 127, put forward, this term might have been chosen for this method precisely because the method made use of coefficients taken from a ’table’ — the one which is now referred to as ‘Pascal triangle’ (see Figure 11, and Chemla (1994)). This table occurred in Yang Hui’s Xiangjie jiuzhang suanfa (Detailed Explanations of The Nine Chapters on Mathematical Methods), in chapter 5, devoted to root extractions, and it probably occurred at the beginning of the chapter. Yang Hui attributes the table to Jia Xian. Here, the term licheng might refer to the table, from the viewpoint of its function of recording in its lines the coefficients required for successive degrees of root extraction. However, a similar table occurs Zhu Shijie’s Siyuan yujian (Jade Mirror of the Four Origins; see Figure 12), this time at the beginning of the book. By contrast with the previous layout of the table, horizontal lines are drawn, and a caption on the left reads: “To extract, one looks at (this) horizontally.” Moreover, oblique lines are also drawn, and they correspond to another reading of the table that plays a key part in Yang Hui’s text related to this table (Chemla 1994). Other captions in Zhu Shijie’s diagram also correspond to the oblique lines. These oblique lines relate to the generation of the table. In this respect, they highlight a relationship between all algorithms for root extractions, and as a result the table accounts for the correctness of the other (and general)

59 The title (and also only the term licheng) of the mathematical part of the writing on the manuscript is repeated several times at the beginning of the verso part of the manuscript (where the mathematical part is written), as if it were a writing exercise. At the end of the table, we read “the mathematical method of licheng licheng suanfa 立成算法” — a part of the text that is not mentioned in Li Yan 李繼 (1954, p. 51). On the interpretation of licheng, see Libbrecht (1982, p. 225).

60 This method for cube root extraction, for instance, is twice quoted by Yang Hui: in chapter 5 of his Detailed Explanations of “The Nine Chapters on Mathematical Methods” (Xiangjie jiuzhang suanfa 講解九章算法 only known through the Yongle dadian, Chapter 16344, pp. 16-17), where the commentary explicitly attributes its formulation to Jia Xian (see fn 3, above); and again in the still extant last chapter of this book, Compilation and Classification of “Detailed Explanations of ‘The Nine Chapters on Mathematical Procedures’” (Xiangjie jiuzhang suanfa zuanlei 講解九章算法纂類), where it is named after Jia Xian (Yongle dadian (Grand Classic of the Yongle Period), chapter 16344, p. 16, and also on pp. 37a-b of the 1842 edition, Guo Shuchun 郭書春 (ed.) (1993), p. 1022). On the differences between the two, see Chemla (1994).
procedure for root extraction, introduced by Jia Xian (Chemla 1994). As far as we can tell from the extant mathematical documents, by contrast to the table-relations encountered previously, in the context of writings, the nature of the table and of its text both seem to be new. Here lies precisely the shift in the texts used to write down table-relations that I announced in the introduction and wanted to establish.

A parallel can, however, be drawn between the Qin computing tool mentioned above and this table: relying on the layout of the table, lines are drawn, in the former case, horizontally and vertically, and in the latter case, horizontally and obliquely. These lines seem to be essential for the functions of each of the tables. In the former case, these lines are the support for a calculational function of the table. In the latter case, these lines are related to the meaning of sequences of numbers they allow the user to select in the table (both horizontally, in relation to root extraction, and obliquely, in relation to the generation of the table). In the former case, the table is an object, in the latter, it is inserted within the pages of a book. The key point for me is that by contrast to Yang Hui, whose commentary refers to the triangle of numbers with the term licheng, Zhu Shijie refers to it as a ‘diagram tu’. I will return to this point below. What is striking is that the table presented in Figure 1 is also referred to as a ‘diagram tu’. Following actors’ categories, I suggest to refer to these texts for tables using the expression ‘diagrammatic table.’

Let us now return to the latter table-layout, which also occurs as part of Yang Hui’s sub-commentary to The Nine Chapters, from 1261. The text of the table is likewise placed at the beginning of the chapter devoted to the right-angled triangle, with three geometrical diagrams (also referred to as tu) that are essential to the theory of that geometrical shape (Chemla 2005). Most probably, like the previous one, the table-layout was part of Jia Xian’s commentary. I argue that, like the three geometrical diagrams, the table-relation that is expressed using a table-layout is likewise essential to the theory developed in this chapter. In the chapter, 13 quantities are attached to any right triangle: these quantities occur in line 1 of the table-layout, and they represent a completion of the theory by comparison with the related chapter of The Nine Chapters (see the translation of the table in Figure 13). The table-relation establishes that each of these 13 quantities (one per column) can be expressed as a function of three fundamental ones among them, and only these three: height minus base (b-a), hypotenuse minus height (c-b) and difference between the hypotenuse and the sum of the height and the base (a+b-c). These three quantities occur as headings of the lines 5 to 7, in the first column. For each of the 13 quantities, the corresponding column records three numbers, m, n, p, in the corresponding lines: these numbers correspond to the formula:

quantity in the column=m.(b-a) + n.(c-b) + p.(a+b-c)
In other words, with these three basic values among the thirteen, all the others can be expressed with coefficients that are natural integers. The clauses of the table-relation are thus columns. They further include numerical values of the 13 quantities for a right triangle whose sides are 8, 15, and 17. The three basic values with respect to which all the others are expressed in the table are the smallest ones (2, 6, 7) for the numerical example chosen. In fact, the algebraic computations that the table embodies show that for any right-angled triangle these three values are the smallest ones (even though the order among them can possibly change). Moreover, the numerical example is chosen in such a way that all values are different, the sums and the differences at a numerical level being different and thus reflecting the sums and differences at the algebraic level, as the algebraic expression by columns makes clear. This seems to be what the next part of the text makes explicit. The remarks on these values also fit with the type of research on numerical values and formulas for the right triangle to which Li Ye’s Ceyuan haijing 测圆海镜 (Sea Mirror of the Circle Measurements; 1247) attests.61

The table-layout enables the author of the table to express 13 formulas in an economic way as columns, and also to highlight the key part played by three fundamental quantities. But there is more. Two columns can be added, or subtracted one from the other (all coefficients of one being greater than all coefficients in the other), and yield new formulas, or yield other extant columns. So again, in this case, horizontal and vertical lines play a key part in expressing the table-relation, and enabling operations on its columns. And, again, this is correlated with the layout chosen for the ‘diagrammatic table.’ Finally, it is interesting that Yang Hui uses the term ‘check/prove yan 驗’ in relation to the intellectual operations that can be carried out on the table. Indeed, Chemla (2010) provides evidence that in the context of the ancient commentaries, the term yan systematically referred to a type of proof based on a visual tool. This suggests that the use of the term tu to refer to the diagrammatic support and the positioning of tables at the beginning of chapters, like fundamental geometric diagrams, have deeper meanings than we might have imagined.

Conclusion: A Historical Shift in the Types of Text Used for ‘Table-Relations’

In this article, I have established that the discursive part of the earliest known mathematical manuscripts was composed of (at least) two types of elements,

61 See Lin Lina 林力娜 (1993). This remark suggests that Li Ye might have used this table, and that it thus yields additional evidence for the conclusion that this layer of the book can be attributed to Jia Xian (see footnote 4).
marked by two types of text. The manuscripts alternate continuous text, and
text for table-relations. In these manuscripts, the latter were written down as
‘textual tables,’ and I have shown that two basic types of styles were used
for these textual tables. By contrast, table-layouts have been used for a Qin
object and a Dunhuang manuscript carrying table-relations. I have sug-
gested to interpret these artifacts as objects that were computing tools. We
need to carry out a closer analysis of the Zhangjiajie document to confirm
these conclusions.

By contrast, it seems that, at least from the eleventh century onwards,
diagrammatic tables were introduced into books. They are used to write
down new types of table-relations. In a sense, both types of tables are placed
at the beginning of books, or of chapters, but this position might have had a
different meaning, depending on the kind of text in which the table occurred.

Moreover, the new type of table is characterized by the fact that dia-
grammatic features like horizontal, vertical and oblique lines play a key part
in the reading, interpretation and use of these table-relations. In this sense,
they can be compared with the Qin computing tool.

The fact that in Song-Yuan times these diagrammatic tables are referred
to as a ‘diagram tu clarsimp’ has a curious echo. Indeed, Chemla (2010) argues that
before the tenth century, mathematical activity to which extant documents
attest used discursive texts, and three types of material objects: counting
rods, blocks for space geometry, and ‘diagram tu clarsimp’ for plane geometry. The
article gives evidence that at the time, the latter were indeed material objects.
It further argues that a shift occurred in the tenth century, both in the
meaning of the term ‘tu clarsimp’ and also in the nature of mathematical writings.
On the one hand, illustrations for all the elements of practice that formerly
were material objects were progressively introduced into books. On the
other, the term ‘tu 手続き’ changed meaning in that it was used to designate all
these types of illustration.

If what I have argued for in this article holds true, it seems that table-
relations followed a similar pattern. Those inserted in the earliest known
manuscripts were textual tables, while those having specific layouts with
diagrammatic features were objects used to operate with them. From the
eleventh century on at the latest, in relation to the shift in the composition of
writings, new types of text for tables took shape and were introduced into
documents on paper. In addition, they were also referred to as ‘tu 手続き.’ I am
aware that these conclusions are tentative, and they might be contradicted
by new discoveries. They nevertheless raise an issue that will remain: the
history of the textualization of tables and its relation with the wider history
of writings and objects used in the context of mathematical activity.
**Figures**

**Figure 1.** Table from chapter ‘Gou gu 勾股 (Right triangle)’ in *Detailed Explanations of “The Nine Chapters on Mathematical Methods”*

Figure 2. Table from *Suanshu* 算術 (Mathematical Procedures)

Figure 3. Qinghua slips “Calculating table” from the Qin dynasty, excerpt

Figure 4. Dunhuang manuscript *Pelliot chinois* 2490, a table for the computation of areas of croplands, written down in 952. The sheets are attached to a pole that can be seen on the left hand side.

SOURCE: By courtesy of the Bibliothèque Nationale de France.
Figure 5. Multiplication table from the manuscript *Suan jing* 算經 (Mathematical Canon), *Pelliot chinois* 3349

SOURCE: Copyright Bibliothèque Nationale de France. Reproduced by courtesy of the Bibliothèque Nationale de France.
Figure 6. Licheng suanjing (Mathematical Canon with Tables), Dunhuang manuscript Or. 8210/5.930

SOURCE: Copyright Or.8210/5.930 British Library Board. Reproduced by courtesy of the British Library.
Figure 7. Dunhuang manuscript Or.8210/S.19

SOURCE: Copyright Or.8210/S.19 British Library Board. Reproduced by courtesy of the British Library.
Figure 8. The Nine Chapters on Mathematical Procedures (Jiuzhang suanshu). First part of the table of equivalences among grains, which is written using two registers.

SOURCE: Shanghai tushuguan and Beijing daxue tushuguan (1980), Songke suanjing liu zhong (Six Mathematical Canons Printed During the Song), The Nine Chapters, chapter 2, p. 1.
Figure 9. *The Gnomon of the Zhou (Zhoubi)*. First part of the table of lengths of gnomon shadows at various moments of the year. The text of the table records a single clause per column.

<table>
<thead>
<tr>
<th>Time</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winter</td>
<td>長一丈三尺五寸</td>
</tr>
<tr>
<td>Spring</td>
<td>長一丈五寸四分一小分</td>
</tr>
<tr>
<td>Summer</td>
<td>長二尺五寸小分</td>
</tr>
<tr>
<td>Autumn</td>
<td>長二尺五寸小分</td>
</tr>
</tbody>
</table>

Figure 10. *Mathematical Classic by Master Sun (Sunzi suanjing).* Multiplication table, on the basis of which computations are grafted.

SOURCE: Shanghai tushuguan and Beijing daxue tushuguan (1980), *Songke suanjing liu zhong* (Six Mathematical Canons Printed During the Song), *Mathematical Classic by Master Sun (Sunzi suanjing)*, chapter 1, p. 5b.
**Figure 11.** Table from chapter on root extractions, Yang Hui’s 1261 *Detailed Explanations of “The Nine Chapters on Mathematical Methods.”* Yang Hui attributes it to the eleventh-century scholar Jia Xian.

**SOURCE:** *Xiangjie jiuzhang suanfa*, encyclopedia *Yongle dadian* (chapter 16344, p. 5b-6a), Reprint Guo Shuchun (ed.) (1993), p. 1416.
Figure 12. The ‘Diagram for the eighth power according to the ancient method,’ at the beginning of Zhu Shijie’s *Siyuan yujian* (Jade Mirror of the Four Origins), 1303.

Figure 13. Translation of Figure 1, i.e. Table from chapter ‘Gou gu’ (Right triangle) in Detailed Explanations of ‘The Nine Chapters on Mathematical Methods’.

"Diagram of the thirteen names that generate transformations in ‘base-height’ (i.e., right triangle). Base, height and hypotenuse, when added (to each other) make sums and when subtracted (from each other) make differences. When they are equal, then this makes transformations, this makes pieces. When they are multiplied by each other, this makes number-products; this makes areas; if they are of utility, one picks them up. When they have no utility, one does not pick them up. One establishes a diagram to prove/check (yan) these transformations.

<table>
<thead>
<tr>
<th>c - c</th>
<th>c + c</th>
<th>b c +</th>
<th>a c +</th>
<th>a b +</th>
<th>b c -</th>
<th>a c -</th>
<th>a b -</th>
<th>hypotenuse (hyp. or c)</th>
<th>height (b)</th>
<th>base (a)</th>
<th>Explanation of names</th>
</tr>
</thead>
<tbody>
<tr>
<td>From hyp., one subtracts difference between height and base</td>
<td>base height with hyp.</td>
<td>base height with hyp.</td>
<td>height with base from height</td>
<td>height with base from hyp.</td>
<td>height with hyp.</td>
<td>base subtracted from hyp.</td>
<td>base subtracted from height</td>
<td>oblique (between two opposite corners of the field)</td>
<td>length of the rectangular field</td>
<td>width of the rectangular field</td>
<td>Values assumed</td>
</tr>
<tr>
<td>Three pieces</td>
<td>One piece</td>
<td>Nine pieces</td>
<td>Five pieces</td>
<td>Seven pieces</td>
<td>Five pieces</td>
<td>One piece</td>
<td>Two pieces</td>
<td>One piece</td>
<td>Four pieces</td>
<td>Three pieces</td>
<td>Two pieces</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>40</td>
<td>24</td>
<td>32</td>
<td>25</td>
<td>23</td>
<td>2</td>
<td>9</td>
<td>7</td>
<td>17</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>100</td>
<td>36</td>
<td>1600</td>
<td>576</td>
<td>1024</td>
<td>625</td>
<td>529</td>
<td>4</td>
<td>81</td>
<td>49</td>
<td>289</td>
<td>225</td>
</tr>
</tbody>
</table>

The base being eight chi, the height fifteen chi, one asks how much the hypotenuse makes.
Answer: seventeen chi.

Explanation of the problem: in the original problem, the base was three, the height was four, and one looked for the hypotenuse; five. The corresponding differences between the values being one, they could not enable us to check/prove the method. Here, we rely on the values of a subsequent problem(s) to explain this (matter) clearly. The shape is like half a triangular field/figure."
References

Traditional Works in Eastern Asian Languages:

Licheng suanjing 立成算經 (Mathematical Canon with Tables), verso of Dunhuang manuscript Or.8210/S.930, British Library, available online at: http://idp.bnf.fr/database/oo_scroll_h.a4d?uid=13145407506; recnum=929;index=1.

Peng Hao 彭浩 (2001), Zhangjiashan Hanjian «Suanshu shu» zhushi 張家山漢簡《算數書》注釋 (Commentary on Writings on Mathematical Procedures, a Document on Bamboo Strips Dating from the Han and Discovered at Zhangjiashan), Beijing: Kexue chubanshe 科學出版社.

Qian Baocong 錢寶琮 (1963), Suanjing shishu (Qian Baocong jiaodian) 算經十書 (錢寶琮校點) (The Ten Classics of Mathematics (Critical Punctuated Edition)), Beijing: Zhonghua shuju 中華書局.

Qinghua daxue chutu wenxian yanjiu yu baohu zhongxin 清華大學出土文獻研究與保護中心 (Qinghua University Center for Research and Protection of Excavated Documents) and Li Xueqin 李學勤 (2013), Qinghua daxue cang Zhangguo zhujian (si) shangce, xiace 清華大學藏戰國竹簡 (肆) 上冊丶下冊 (Bamboo Slips from the Warring States Period Kept at Qinghua University. Part 4, first and second volumes), Shanghai: Zhongxi shuju 中西書局.

Shanghai tushuguan 上海圖書館 (Shanghai Library) and Beijing daxue tushuguan 北京大學圖書館 (Peking University Library) (1980), Songke suanjing liu zhong 宋刻算經六種 (Six Mathematical Canons Printed During the Song), Beijing: Wenwu chubanshe 文物出版社.


Suan jing 算經 (Mathematicical Canon), Dunhuang manuscript Pelliot chinois 3349, Bibliothèque Nationale de France, available online at http://idp.bnf.fr/database/oo_scroll_h.a4d?uid=129241475011;recnum=60642;index=4.
 Untitled, Dunhuang manuscript Or.8210/S.19, British Library, available online at: http://idp.bnf.fr/database/oo_scroll_h.a4d?uid=13881787816;recnum=18;index=1.


Zhu Hanmin 朱漢民 and Chen Songchang 陳松長 (gen. eds.) (2011), Yuelu shuyuan cang Qin jian (er) 嶽麓書院藏秦簡（貳） (Qin Bamboo Slips Kept at the Academy Yuelu (2)), Shanghai: Shanghai cishu chubanshe 上海辭書出版社.

Secondary Sources in Western and Eastern Languages


Chôka zan kankan Sansûsho kenkyûkai 張家山漢簡『算數書』研究会 (Research Group on the Han Bamboo Strips from Zhangjiashan Writings on Mathematical Procedures) (2006), Kankan Sansûsho 漢簡『算數書』(The Han Bamboo Strips from Zhangjiashan Writings on Mathematical Procedures), Kyoto: Hôyû shoten 朋友書店.


Karine Chemla: Numerical Tables in Chinese Writings


Han Wei 韓巍 (2012), “Beida Qin jian zhong de shuxue wenxian” 北大型簡中的數學文獻 (Mathematical Documents from the Qin Slips of Peking University), Wenhui 文化 (Cultural Relics) 2012. 6: 85-89.


Hubei sheng wenwu kaogu yanjiusuo 湖北省文物考古研究所 (Research Institute for Cultural Relics and Archaeology of Hubei Province) and Yunmeng xian bowuguan 雲夢縣博物館 (Museum of Yunmeng District) (2008), “Hubei Yunmeng Shuihudi M77 fajuejianbao” 湖北雲夢睡虎地 M77 發掘簡報 (A Concise Report on the Excavation of the Tomb M77 at
Shuihudi, in Yunmeng, Hubei), *Jiang Han kaogu* 江漢考古 (Jiang Han Archeology) 109. 4: 31-37 and Plates 11-16.


Li Junming 李均明 and Feng Lisheng 冯立昇 (2013), “Qinghua jian ‘Suan biao’ gaishu” 清華簡《算表》概述 (Overview of the Slips “Numerical Table” Kept at Qinghua University), *Wenwu* 文物 (Historical Relics) 2013. 8: 73-75.


Mo Zihan 墨子涵 (Daniel Morgan) and Lin Lina 林力娜 (Karine Chemla) (2016), “Ye you lunzhe xie de: Zhangjiashan Hanjian ‘Suanshu shu’ xieshou yu pian xuchuan” 也有論著的：張家山漢簡《算數書》寫手與篇序初探 (There is Also Writing in Turns: Initial Investigation of the Hands and Compilational Order of the Han Bamboo Manuscript Suanshu shu (Writings on Mathematical Procedures) from Zhangjia-shan), Jianbo 筆帛 (Bamboo and Silk Manuscripts) 12: 235-252. A revised English version is forthcoming in Silk and Bamboo, 1, 2017.


―― (2015), Yuelu shuyuan cang Qin jian “Shu” yanjiu 岳麓書院藏秦簡“數”研究 (Research on the Qin Manuscript on Bamboo Slips Mathematics Kept at the Academy Yuelu), Beijing: Zhongguo shehui kexue chubanshe 中國社會科學出版社.