Tina Su Lyn Lim and Donald B. Wagner, *The Continuation of Ancient Mathematics: Wang Xiaotong's* Jigu suanjing, *Algebra and Geometry in 7th-Century China*, Copenhagen: NIAS Press, 2017, xii+220 pp.

Jia-Ming Ying

[Jia-Ming Ying is an Assistant Professor in the Department of Mathematics and Information Education, National Taipei University of Education, Taipei. He received his Ph.D. from the Department of Mathematics, National Taiwan Normal University, for the thesis "Nam Pyŏng-Gil's (1820-69) Reinterpretation of Classical Mathematics" (written in traditional Chinese). He was a Li Foundation Scholar at the Needham Research Institute, Cambridge, and a former faculty member of the Center for General Education, Taipei Medical University, Taipei. Contact: j.m.ying@tea.ntue.edu.tw]

Introduction

For decades Professor Donald B. Wagner has studied and published scholarly works about general East Asian history and especially the history of mathematics, science, and technology in pre-modern China. For the part of Chinese mathematics, he has made significant contributions ever since his works in 1970s about Liu Hui's 劉徽 (third century) commentaries on the volumes of a pyramid and the sphere (Wagner 1978, Wagner 1979). In 2017, Professor Wagner and Tina Su Lyn Lim, a student of his in the University of Copenhagen, published this book about their analysis of the text *Jigu suanjing* 緝古算經 (Continuation of Ancient Mathematics, ca. 626) written by Wang Xiaotong 王孝通 (late sixth century to mid-seventh century).

This book is one of the few monographs, if not the only one, in a Western language in modern times that is devoted to the discussion of mathematics in the Tang 唐 dynasty (618-907). In this reviewer's opinion, there are three main periods of pre-modern Chinese history that draw much attention of historians of mathematics due to the abundance of mathematical sources and/or developments in those times. The first period is the early imperial China until the third century, and the focus of research is on the contents and commentaries of several mathematical classics, especially about the *Jiuzhang suanhshu* 九章算術 (Nine Chapters of Mathematical Art, first century) and Liu Hui's commentaries; the second is the period of Song 宋, Jin 金 and Yuan 元 Dynasties in the tenth to the fourteenth centuries, and the focus is often on the development of Chinese algebraic methods; the third period is time after the late sixteenth century, and the focus is on the interactions between Chinese and European mathematics. Scholarly discussions in Western languages about mathematics in Tang dynasty, such as Siu and Volkov (1999), are usually about its system of mathematics education instead of the mathematics *per se*. Therefore, this work by Lim and Wagner fills right into the long gap between the times of early-imperial China and the Song dynasty.

It is true that there are not as many mathematicians who made significant contributions that we know of in the Tang dynasty as in some other periods of Chinese history, and that the Tang state education system and their compilation of mathematical textbooks is one of the major reasons that many of the earlier mathematical texts survived to later eras and inspired generations of mathematicians, but it does not mean that there are not some interesting innovations of mathematics that can be discussed during the Tang dynasty. One of the strengths of this work by Lim and Wagner is exactly this: it discusses some unique methods developed by Wang Xiaotong in relation to the Chinese mathematics before and after his time. Besides, this work might interest a range of readers if they take a look at the contents, which are going to be introduced in the next section.

Contents and Structure of the Book

This book, as the title suggests, is about Wang Xiaotong's *Jigu suanjing*, but it does not only provide a translation of the text itself (which is certainly important). The entire Chinese text with notes of small variants in different versions, the full English translation, and the authors' mathematical comments mainly in the form of formulae and equations, are arranged in a format that is easy for the reader to digest in **Part III: Chinese text and translation** of this book. However, many of the reasons that this reviewer believes that the book might delight a range of readers can be found in its **Part I: Background**. Occupying less than one-fifth of the main text, Part I gives the readers general historical and mathematical backgrounds for Wang Xiaotong's work and Chinese mathematics. Even if one does not have much prior knowledge about Chinese mathematics, it is still quite interesting to read about these backgrounds. An historian or an education scholar could be interested in the Tang mathematics education system, while a mathematician might like to read how pre-modern Chinese extract roots of equations.

Part I starts with a brief introduction to Wang Xiaotong and his book, but in the following sections the authors will come back with more details. Since there is not much known about Wang Xiaotong's life, the authors present their estimations about Wang's birth and death years, and about his positions and ranks in the Tang official system. All of these give readers a pretty good idea about when Wang lived and when the text entered the official education system. A reader who is more familiar with the history of the Tang Empire might find that early in the book there is some rather common misinformation about the number of ranks of the Tang official system (p. 3 and p. 13). In fact, the total number of ranks of Tang's official system is 30, not 36, and Wang's position when he presented the text to the throne, Assistant to the Grand Astrologer (*taishi cheng* 太史丞), is of the 22nd rank (*Xin Tang Shu*, vol. 47; *Jiu Tang Shu*, vol. 42). But these numbers would not affect the reading of the main text.

After the brief introduction, Section 1.2 talks about public works planning in ancient China. This is one of the major applications of mathematics in pre-modern East Asia. The authors give examples of calculations of labour requirements in the *Jiuzhang suanshu*, in the *Jigu suanjing*, and in a fourteenth century text to show the applications in different situations and time periods, and they also briefly mention the differences in their mathematical methods, to which they will come back in later sections. Section 1.3 covers the current state of what modern historians know about Tang's mathematics education system. Here the authors explain how the system worked and what textbooks the students studied, including the famous but lost text *Zhui shu* 綴術 by Zu Chongzhi 祖沖之 (429-500). (In discussing the topic, the book quotes the *Sui shu* 隋書 [History of the Sui dynasty], but the character for Sui is misprinted both on p. 16 and in the index.) It is important to discuss this text because in one of the two tracks for the course of mathematics, the *Jigu suanjing* and the *Zhui shu* are the only two textbooks. The authors give a very interesting hypothesis of the contents of the *Zhi shu*. They believe that since the *Jigu suanjing* assumes in the reader a background in the techniques covered by the classic *Jiuzhang suanshu*, the *Zhui shu* should have been an extensively annotated edition of the *Jiuzhang suanshu*, and it should also have contained 'Horner's method' for extracting the roots of polynomials, the procedure that the *Jigu suanjing* uses many times but never explains. With this hypothesis, the combination of the *Jigu suanjing* and the *Zhui shu* can be seen as a concentrated course in advanced practical mathematics.

While the general historical backgrounds in the first three sections would interest readers with a taste for history, even if they have no relevant mathematical knowledge, Sections 1.4 and 1.5 would certainly attract the attention of those with heavier inclinations towards mathematics. Section 1.4 is about the mathematical background that a reader needs to understand the *Jigu suanjing*. In particular, the authors introduce the Chinese rod-counting system, methods about right-angled triangles, and dissections of solids. Section 1.5 goes on to discuss 'Horner's method', the procedure for extracting roots of polynomial equations, which is likely to have been well known to Chinese mathematicians in Wang's time. Beside these topics in the main text, the book also provides several 'information boxes' throughout the main text. Readers who already have enough background or who are not so interested in the technical details can ignore them, but for those who do like mathematics, these 'boxes' are interesting diversions. Box 1 in section 1.4 introduces counting-rod numerals and arithmetic. Box 2 defines a 'Wang Xiaotong cubic function' ($f(x) = x^3 + ax^2 + bx - c$, where $a, b \geq 0$ and $c > 0$) and proves that it has exactly one positive root.

Section 1.6 is about the history of the text of the *Jigu suanjing*. The authors give a short account of how different versions of the text survive to this day. This concludes Part I. As can be seen, readers from different backgrounds can choose the sections they need or are interested in before reading about the technical contents of the *Jigu suanjing*.

Part II: The Mathematics of the *Jigu suanjing*, as the title suggests, analyses the mathematical techniques used by Wang Xiaotong. The order of this Part does not follow the order of the problems in the original text, but goes from the simplest problems to more complicated ones according to the authors' judgements. Sections 2.1 to 2.7 (with the exception of Section 2.2) are problems about different solids that could be applied to the planning of public works: truncated square pyramids and cones, parallelepipeds and cylinders, a wedge, the Grand Astrologer's platform and ramp, a dyke, and a 'dragon tail' dyke. The problems are not simply about calculating the volumes of the solids given the dimensions. The units of measure and the volumes of standard solids in Chinese mathematics are discussed in other classics, such as the *Jiuzhang suanhshu*, and for the reader's information, the authors have included Boxes 3 and 4 in Section 2.1 to explain these basics that were essential to the intended readership of the Tang text. Most of the problems in Wang Xiaotong's text are in fact about finding the unknown dimensions of a solid given certain relations among some known dimensions.

Section 2.1 introduces problems on truncated pyramids and cones, all of whose calculations lead to one or more Wang Xiaotong cubic equations. A reader who has the patience to go through all the details of the analysis given by the authors for these problems will probably find their explanations clear and easy to understand because they do not just provide translations but also diagrams of area and volume dissections that help reasoning. Of course, the authors supply these calculations and reasoning with their interpretations of how Wang Xiaotong arrived at his results. The methods Wang Xiaotong used to obtain his cubic equations, according to the authors, entailed the use of **similar triangles**, **volume dissections**, and **'reasoning about calculations'**, which are discussed in Section 2.2.

The consideration of **similar triangles** in Chinese mathematics is a topic not universally agreed upon among historians of Chinese science. Some scholars argue that traditional Chinese mathematics did not exactly compare the similarity of two triangles (e.g. Cullen (2007)), but they might agree with the authors that Chinese mathematicians of that time did consider the proportional relations of the corresponding sides of two triangles to help them solve problems. Either way, the authors mention in Section 2.2 that Wang Xiaotong never made his use of similar-triangle considerations explicit, and the readers can judge by themselves whether the Tang mathematician actually considered it.

The method of **volume dissections** is, in this reviewer's opinion, one of most interesting methods in Chinese mathematics. Wang Xiaotong often needed some 'intermediate quantity' to arrive at his cubic equations. For instance, there is an 'area for the corner *Yangma*' in Problem 7 introduced in Section 2.1. This quantity does not correspond to any 'area' in the problem. However, from the name of the quantity, the authors give a plausible explanation of how the solid in question is dissected to produce the intermediate quantity and finally of how Wang arrived at his cubic equations. This reviewer strongly recommends the reader go through the works of at least one problem in the book to appreciate the brilliance of the reasoning. Moreover, Section 2.2 also explains how volume dissections could be communicated with scale models or 'chessmen' for a teacher of Chinese mathematics, since no diagrams of three-dimensional geometric situations are found in Wang's or earlier mathematical texts in China.

The authors use the term **'reasoning about calculations'** to avoid using 'algebra' directly, because they believe there is no consensus among scholars as to in what sense do pre-modern Chinese have 'algebra'. They then explain that from the examples in Section 2.1, Wang Xiaotong manipulated the terms in the verbal formulae very much as how we would manipulate those in an equation. It is further emphasised that giving names to intermediate quantities could simplify the statements of calculations, but Wang Xiaotong gave names to *abstract* quantities that do not correspond to any element in the solid, such as the 'area for the corner *Yangma*' mentioned earlier. These abstract quantities, in the authors' judgment, represent a new level of abstraction in Chinese mathematics.

From Section 2.3 through 2.7, the book goes on to discuss more complicated problems and the reasoning behind the equations. The authors try to show the volume dissections, implicit assumptions, and some attractive alternative explanations proposed by themselves or by other scholars. Sections 2.8 and 2.9 discuss another group of problems related to right-angled triangles and the Pythagorean triples. Although the problems are about right-angled triangles, which are two-dimensional shapes, Wang Xiaotong's reasoning, according to the authors, still uses solids to construct his cubic equations in many cases. The authors also discuss reconstructions of some of these problems by the Chinese mathematician Zhang Dunren

張敦仁 (1754-1834) and the Korean mathematician Nam Pyŏng-Gil 南秉吉 (1820-1869).

The final Section of Part II, Section 2.10, returns to the problem of 'reasoning about calculation' and 'algebra'. The authors themselves give a very broad definition of **algebra**: "comprising all the methods which have been used to solve problems which we in modern times solve using algebra." Whether the readers agree with the definition or not, the authors then argue convincingly that Wang Xiaotong's comments show "a recognition that the arrangement of numbers on the counting board used in extracting roots of polynomials represents what we call an equation," because it is "a statement that performing certain operations on an unknown quantity results in a certain known quantity." What this section discusses and some of the mathematical explanations in previous sections constitute, in this reviewer's opinion, the major contribution of the book.

Major Contribution of the Book: Discussions about a Middle Stage of Development in Chinese 'Algebra'

The discussion of Wang Xiaotong's mathematical methods as a developmental stage of Chinese algebra is what this reviewer believes to be the major contribution of the book. As is well known to historians of Chinese mathematics, the algebraic methods developed in early imperial China, represented by certain contents in the *Jiuzhang suanshu* and Liu Hui's commentaries in the third century, include the solution of systems of linear equations and the extraction of square and cube roots (e.g. Chemla and Guo (2004)). Later in the works of the thirteenth century, Chinese mathematicians demonstrated how they could form and manipulate polynomials to arrive at an arrangement of numbers that can be seen as an 'equation' (e.g. Martzloff (1997)). But what happened to Chinese 'algebra' in the thousand years in between? As mentioned earlier, there are few surviving mathematical texts that can tell even a fraction of the story, and Wang Xiaotong's text is one of these few.

The *Jigu suanjing* certainly deals with more complicated problems on volume calculations and public works than the *Jiuzhang suanshu*, and Wang's reasoning about calculation arrived at more general forms of cubic equations, but as the authors show in detail in their translation and comments for the solutions to many problems, Wang Xiaotong did not obtain his cubic equations simply by an algebra of polynomials much as the thirteenth century mathematicians would do or as what modern mathematicians would do. Chinese mathematicians after the thirteenth century would likely set up an unknown, line up several arrangements of numbers according to the statements in the problem, and then *directly* *manipulate* these arrangements by addition, subtraction, multiplication and sometimes simple division, to arrive at the final 'equation'. The authors put Box 5 in Section 2.10 to show this process of manipulations of polynomials with Zhang Dunren's solution in 1803 for one of the problems in the *Jigu suanjing*.

On the other hand, Wang Xiaotong did not do that. He did not seem to directly manipulate polynomials. In his solutions to many of the problems, Wang Xiaotong defined some (abstract or actual) intermediate quantities which were likely obtained by his volume dissections, and then he arrived at his cubic 'equations' by recombining the dissected parts or by reasoning with calculations. It is the belief of this reviewer that the most important contribution of this book is that the authors tell a good mathematical story about a middle stage of development in Chinese 'algebra'. For most problems in Wang's text, the authors give clear diagrams for volume dissections and detailed mathematical explanations for reasoning of calculations. A reader who goes through the process of at least one solution to a problem would likely be amazed by Wang's brilliance and by the authors' clarity in their explanations. Although there is still much that is unknown to modern scholars, Wang Xiaotong's method for forming his cubic equations is certainly one important piece of the big jigsaw puzzle of the development of Chinese 'algebra', and we have Lim and Wagner to thank for showing us this piece.

The Verdict

It is this reviewer's belief that this book might attract a range of readers with different interests or tastes. For sinologists or historians, this book might serve as a gateway to the history of Chinese mathematics. The background information it provides, such as units of measure, arithmetic using counting rods, public work planning, and Tang mathematical education, are all necessary prerequisites for a reader to understand the main themes in this book. And for those who have studied histories of East Asian mathematics for some time, such as myself, reading about the middle stage of development of Chinese algebra is exciting. Moreover, a more mathematically inclined reader will not be disappointed, because the authors discuss the mathematics in the *Jigu suanjing* both from the modern perspective, such as proving the uniqueness of the positive solution of Wang Xiaotong cubic, and from Wang's perspective, for the book gives detailed accounts of how problems could be solved with the mathematical concepts and devices available to Tang dynasty mathematicians. Reading through the process of solving each problem will feel like an adventure. Finally, another less obvious potential group of readers is mathematics educators. Although Wang Xiaotong's reasoning of calculation might not exactly fit in with how we teach high school algebra today, his approaches to volume dissection could be integrated into high school geometry as a way of improving students' three-dimensional thinking. With the help of the authors' clear explanations, a mathematics educator may transform some of the problems and solutions into teaching materials. In summary, the broad readership and the in-depth discussion of Chinese 'algebra' are the two main reasons this reviewer would like to recommend this book.

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